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# GAP ESTIMATES OF THE SPECTRUM OF HILL'S EQUATION AND ACTION VARIABLES FOR KdV

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Abstract. Consider the Schrödinger equation $-y'' + Vy = \lambda y$ for a potential $V$ of period 1 in the weighted Sobolev space ($N \in \mathbb{Z}_{\geq 0}$, $\omega \in \mathbb{R}_{\geq 0}$)

$$H^{N,\omega}(S^1;\mathbb{C}) := \{f(x) = \sum_{k=-\infty}^{\infty} \hat{\hat{f}}(k) e^{i2\pi kx} \;\Big|\; \| f \|_{N,\omega} < \infty\}$$

where $\hat{\hat{f}}(k)$ ($k \in \mathbb{Z}$) denote the Fourier coefficients of $f$ when considered as a function of period 1,

$$\| f \|_{N,\omega} := \left( \sum_k (1+|k|)^{2N} e^{2\omega|k|} | \hat{\hat{f}}(k)|^2 \right)^{1/2} < \infty,$$

and where $S^1$ is the circle of length 1. Denote by $\lambda_k \equiv \lambda_k(V)$ ($k \geq 0$) the periodic eigenvalues of $-\frac{d^2}{dx^2} + V$ when considered on the interval $[0,2]$, with multiplicities and ordered so that $Re\lambda_j \leq Re\lambda_{j+1}$ ($j \geq 0$). We prove the following result.

**Theorem.** *For any bounded set $\mathcal{B} \subseteq H^{N,\omega}(S^1;\mathbb{C})$, there exist $n_0 \geq 1$ and $M \geq 1$ so that for $k \geq n_0$ and $V \in \mathcal{B}$, the eigenvalues $\lambda_{2k}, \lambda_{2k-1}$ are isolated pairs, satisfying (with $\{\lambda_{2k}, \lambda_{2k-1}\} = \{\lambda_k^+, \lambda_k^-\}$)*

(i) $\sum_{k \geq n_0} (1+k)^{2N} e^{2\omega k} |\lambda_k^+ - \lambda_k^-|^2 \leq M$,

(ii) $\sum_{k \geq n_0} (1+k)^{2N+1} e^{2\omega k} \left| (\lambda_k^+ - \lambda_k^-) - 2\sqrt{\hat{\hat{V}}(k)\hat{\hat{V}}(-k)} \right|^2 \leq M$.

## 1. Introduction and summary of the results

The Korteweg-deVries equation (KdV) on the circle

$$\partial_t U(x,t) = -\partial_x^3 U(x,t) + 6U(x,t)\partial_x U(x,t) \tag{1.1}$$

is a completely integrable Hamiltonian system of infinite dimension. We choose as its phase space the Sobolev space $H^{N,\omega}(S^1)$, where $S^1$ is the circle of length $1, \omega \in \mathbb{R}_{\geq 0}$ and $N \in \mathbb{Z}_{\geq 0}$. The Poisson structure is the one proposed by Gardner,

$$\{F_1, F_2\}_G := \int_{S^1} \frac{\partial F_1}{\partial V(x)} \frac{d}{dx} \frac{\partial F_2}{\partial V(x)} dx,$$

where $F_1$ and $F_2$ are $C^1$ functionals on $H^{N,\omega}(S^1)$ and $\frac{\partial F}{\partial V(x)}$ denotes the $L^2$-gradient of $F$. The Gardner bracket is degenerate. Its symplectic leaves are given by

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$H_c^{N,\omega}(S^1) := c + H_0^{N,\omega}(S^1)$ with $c \in \mathbb{R}$, where

$$H_0^{N,\omega}(S^1) := \{f \in H^{N,\omega}(S^1) | \int_{S^1} f dx = 0\}.$$

Also we introduce the following weighted $l^2$-spaces:

$$l^2_{N+\frac{1}{2},\omega}(\mathbb{N};\mathbb{R}^2) := \{(x,y) = (x_j, y_j)_{j\geq 1} | \sum_{j\geq 1} j^{2N+1} e^{2\omega j}(x_j^2 + y_j^2) < \infty\}.$$

In section 3 we prove the following result.

**Theorem 1.** *Let $N \in \mathbb{Z}_{\geq 0}$ and $\omega \in \mathbb{R}_{\geq 0}$. Then there exists a map*

$$\Lambda^{(N,\omega)} : H_0^{N,\omega}(S^1) \to l^2_{N+\frac{1}{2},\omega}(N;\mathbb{R}^2)$$

*with the following properties:*

(1) *$\Lambda^{(N,\omega)}$ is a diffeomorphism;*
(2) *$\Lambda^{(N,\omega)}$ and $(\Lambda^{(N,\omega)})^{-1}$ are real analytic;*
(3) *the variables $(x_j^2 + y_j^2)/2$ associated to $\Lambda^{(N,\omega)}(V) = (x_j(V), y_j(V))_{j\geq 1}$ are action coordinates of KdV and its entire hierarchy.*

We refer to $(x_j(V), y_j(V))_{j\geq 1}$ as Birkhoff coordinates of KdV (and its hierarchy). The map, associating to $V$ Birkhoff coordinates, is referred to as Birkhoff map, and can be thought of as a nonlinear Fourier transform. Clearly, the Fourier transform $\mathcal{F}$ establishes a linear isomorphism between $H_0^{N,\omega}(S^1)$ and $l^2_{N,\omega}(\mathbb{N};\mathbb{R}^2), \mathcal{F}(V) = (Re\hat{\hat{V}}(k), Im\hat{\hat{V}}(k))_{k\geq 1}$, and Theorem 1 is an instance of (probably) many properties Fourier transform and Birkhoff map have in common. Theorem 1 has already been established in the case $\omega = 0$ [BKM1] (cf. also [Ka], [BBGK]). In order to prove that $\Lambda^{(N,\omega)}$ can be chosen as the restriction of $\Lambda$ to $H_0^{N,\omega}$, one has to derive asymptotic estimates for the periodic eigenvalues of the Schrödinger operator $L := -\frac{d^2}{dx^2} + V$ for $V$ in $H_0^{N,\omega}(S^1;\mathbb{C})$ considered on the interval [0,2]. The periodic spectrum of $L$ is discrete. Denote it by $(\lambda_k = \lambda_k(V))_{k\geq 0}$ (with multiplicities), where the $\lambda_k$'s are ordered in such a way that $Re\lambda_0 \leq Re\lambda_1 \leq \dots$ and, in case $Re(\lambda_k) = Re(\lambda_{k+1})$, $Im\lambda_k \leq Im\lambda_{k+1}$. For $k$ sufficiently large, the eigenvalues come in isolated pairs $\{\lambda_{2k}, \lambda_{2k-1}\}$. The main result of this paper is the following one, proved in section 2:

**Theorem 2.** *Let $\mathcal{B} \subseteq H^{N,\omega}(S^1;\mathbb{C})$ be a bounded set of potentials ($N \in \mathbb{Z}_{\geq 0}$, $\omega \in \mathbb{R}_{\geq 0}$). Then there exists $n_0 \geq 1$ such that*

$$\sup_{V\in\mathcal{B}} \sum_{k\geq n_0} (1+k)^{2N+1} e^{2\omega k} \left| \lambda_k^+(V) - \lambda_k^-(V) - 2\sqrt{\hat{\hat{V}}(k)\hat{\hat{V}}(-k)} \right|^2 < \infty,$$

*where $\{\lambda_k^+, \lambda_k^-\} = \{\lambda_{2k}, \lambda_{2k-1}\}$ (cf. Theorem 2.10 for the indexing $\lambda_k^+, \lambda_k^-$ of the numbers $\lambda_{2k}, \lambda_{2k-1}$).*

As a consequence of Theorem 2 one obtains

**Corollary 3.** *For a real valued potential $V \in L^2(S^1)$ to be an element of $H^{N,\omega}(S^1)$ it is necessary and sufficient that*

$$\sum_{k=1}^{\infty} (1+k)^{2N} e^{2\omega k} (\lambda_{2k}(V) - \lambda_{2k-1}(V))^2 < \infty.$$

In the case $\omega = 0$, Theorem 2 and Corollary 3 have been established by Marčenko [Ma] by different methods which might, however, not be adaptable to the case $\omega > 0$.

Theorem 1 can be used to prove by the 'inverse scattering method' that the Korteweg-deVries equation (1.1) is well-posed on the circle. To simplify the wording of the statement we restrict ourselves to the case where the initial data $V$ is in $H_0^{N,\omega}(S^1)$ (cf. [BKM2] in case $V$ has nonzero average).

**Corollary 4.** *Let $N \in \mathbb{Z}_{\geq 0}$ and $\omega \in \mathbb{R}_{\geq 0}$. There exists a solution operator $\mathcal{S}$ : $H_0^{N,\omega}(S^1) \to C(\mathbb{R}; H_0^{N,\omega}(S^1))$ of (1.1) with the following properties:*

(i) *Given $V_1, V_2$ in $H_0^{N,\omega}(S^1)$, there exists $M > 0$ so that for any $t \in \mathbb{R}$*

$$\| \mathcal{S}(V_1)(t) - \mathcal{S}(V_2)(t) \|_{H_0^{N,\omega}(S^1)} \leq M(1 + |t|) \| V_1 - V_2 \|_{H_0^{N,\omega}(S^1)} .$$

(ii) *For any $0 < T < \infty$, $\mathcal{S} : H_0^{N,\omega}(S^1) \to C([-T, T]; H_0^{N,\omega}(S^1))$ is real analytic.*

*Proof.* The case $\omega = 0,\ N = 0$ can be treated as in [BKM2] (cf. also [Bo]). The same proof works for this more general situation. In fact, the case $\omega = 0,\ N \geq 1$ or $\omega > 0,\ N \geq 0$ is somewhat easier, as the frequencies of the KdV Hamiltonian are easily seen to be real analytic in these cases. □

*Remark.* Results similiar to the one presented for KdV hold for any of the equations in the KdV hierarchy.

## 2. Proof of Theorem 2

In this section, we prove Theorem 2, stated in the introduction. First let us introduce some more notation.

**Definition.** $w := (w(k))_{k\in\mathbb{Z}}$ is said to be a weight if

(i) $w(k) \geq 1$ $(k \in \mathbb{Z})$;
(ii) there exists $M_w \geq 1$ such that $w(k) \leq M_w w(k - j)w(j)$ $(k, j \in \mathbb{Z})$.

Condition (ii) is refered to as the submultiplicative property of a weight.

Most frequently we will use the weight

$$w(k) := (1 + |\frac{k}{2}|)^N \ e^{\frac{\omega}{2}|k|}, \tag{2.1}$$

where $N \in \mathbb{Z}_{\geq 0}$ and $\omega \in \mathbb{R}_{\geq 0}$. In that case, one can choose $M_w = 1$ in condition (ii) of the above definition. The reason for choosing $\frac{\omega}{2}$ rather than $\omega$ in (2.1) follows from the observation that

$$V = \sum_k \hat{\hat{V}}(k)e^{i2\pi kx} = \sum_k \hat{V}(2k)e^{i\pi(2k)x}$$

for $V \in H^{N,\omega}(S^1; \mathbb{C})$, with $(\hat{V}(k))_{k\in\mathbb{Z}}$ denoting the Fourier coefficients of $V$ considered as a function of period 2 and thus

$$\| V \|_{N,\omega}^2 = \sum_k (1 + |k|)^{2N} \ e^{2\omega|k|} | \ \hat{\hat{V}}(k)|^2 = \sum_k \left((1 + \frac{|k|}{2})^N \ e^{\frac{\omega}{2}|k|}\right)^2 |\hat{V}(k)|^2 .$$

For $K \subseteq \mathbb{Z}$ and a weight $w$ denote by $l_w(K)$ the complex Hilbert space $l_w^2(K) \equiv l_w^2(K; \mathbb{C})$,

$$l_w^2(K) := \{(a(k))_{k\in K} \mid \ \| a \|_w < \infty\}$$

where

$$\| a \|_w \ \equiv \ \| a \|_{l^2_w(K)} := \Big( \sum_{k \in K} w(k)^2 |a(k)|^2 \Big)^{1/2}.$$

Most frequently, we will use for $K$ the set $\mathbb{Z}$ or $\mathbb{Z}(n) := \mathbb{Z} \setminus \{\pm n\}$. If necessary for clarity, we will sometimes write $a_K$ for a sequence $(a(k))_{k \in K} \in l^2_w(K)$.

For a linear operator $A : l^2_{w_1}(K_1) \to l^2_{w_2}(K_2)$ we denote by $A(k,j)$ its matrix elements

$$(Aa)(k) := \sum_{j \in K_1} A(k,j) a(j) \quad (k \in K_2).$$

**Definition.** $\mathcal{S} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is defined by $(\mathcal{S}a)(k) := a(k+1)$ $(k \in \mathbb{Z})$. $\mathcal{S}$ is called the shift operator. The restriction of $\mathcal{S}$ to $l^2_w(K)$ with values in $l^2_{\mathcal{S}w}(K)$ is denoted by $\mathcal{S}$ as well, and $\mathcal{S}^n = \mathcal{S} \circ \cdots \circ \mathcal{S}$ denotes the $n$th iterate of $\mathcal{S}$. Notice that

$$\begin{aligned} \| \mathcal{S}^n a \|^2_{l^2_{\mathcal{S}^n w}(K)} &= \sum_{k \in K} (\mathcal{S}^n w)(k)^2 \Big| (\mathcal{S}^n a)(k) \Big|^2 \\ &= \sum_{k \in K} w(k+n)^2 \Big| a(k+n) \Big|^2 \quad \le \| a \|^2_{l^2_w(\mathbb{Z})} . \end{aligned} \tag{2.2}$$

**Definition.** $\mathcal{J} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is the involution given by

$$(\mathcal{J}a)(k) := a(-k).$$

To prove Theorem 2, it suffices to consider potentials $V \in H^{N,\omega}_0(S^1)$ (where $S^1$ is the circle of unit length), as adding a constant $c$ to $V$ simply shifts the periodic spectrum of $-\frac{d^2}{dx^2} + V$ by $c$.

Express $-\frac{d^2}{dx^2} + V - \lambda$, acting on functions periodic of period 2, in Fourier space, as $A : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, with

$$A(k,j) = \ \pi^2 k^2 \delta_{kj} \ + \ \hat{V}(k-j).$$

Recall that $\hat{V}(0) = \frac{1}{2} \int_0^2 V(x) dx = 0$, as $V \in H^{N,\omega}_0(S^1;\mathbb{C})$, and that $(\hat{V}(k))_{k \in \mathbb{Z}}$ denote the Fourier coefficients of $V$ when considered as functions of period 2.

To analyze the eigenvalues $\lambda_{2n}, \lambda_{2n-1}$ near $n^2\pi^2$ $(n \ge 1)$, write $\lambda = n^2\pi^2 + z$. Writing $l^2(\mathbb{Z})$ as a direct sum $l^2(\mathbb{Z}) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{Z}(n)$, $a = (a(-n),\ a(n),\ a_{\mathbb{Z}(n)})$ $(\mathbb{Z}(n) := \mathbb{Z} \setminus \{\pm n\})$, we see that $A - \lambda$ is of the form

$$\begin{aligned} &\begin{pmatrix} \Big((A-\lambda)(k,-n)\Big)_{k \in \mathbb{Z}} & \Big((A-\lambda)(k,n)\Big)_{k \in \mathbb{Z}} & \Big((A-\lambda)(k,j)\Big)_{\substack{k \in \mathbb{Z} \\ j \in \mathbb{Z}(n)}} \end{pmatrix} \\ &= \begin{pmatrix} -z & \hat{V}(-2n) & (\mathcal{S}^n \mathcal{J} \hat{V})^T_{\mathbb{Z}(n)} \\ \hat{V}(2n) & -z & (\mathcal{S}^{-n} \mathcal{J} \hat{V})^T_{\mathbb{Z}(n)} \\ (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} & (\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)} & B_n - z \end{pmatrix}, \end{aligned} \tag{2.3}$$

where the superscript $T$ denotes the transpose and where $B_n : l^2(\mathbb{Z}(n)) \to l^2(\mathbb{Z}(n))$ is given by

$$\begin{aligned} B_n &:= A_n - \pi^2 n^2 Id_n, \\ A_n &:= (A_n(j,k))_{j,k \in \mathbb{Z}(n)}. \end{aligned} \tag{2.4}$$

The (possibly) complex number $\lambda = n^2\pi^2 + z$ is a periodic eigenvalue for $-\frac{d^2}{dx^2} + V$ if there exists $a = (a(k))_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that

$$(A - \lambda)a = 0.$$

With $x := a(-n), y := a(n)$, this equation can be written as a system of three equations:

$$-zx + \hat{V}(-2n)y + \langle \mathcal{S}^n \mathcal{J}\hat{V}, a_{\mathbb{Z}(n)}\rangle \quad = 0, \tag{2.5}$$

$$\hat{V}(2n)x - zy + \langle \mathcal{S}^{-n}\mathcal{J}\hat{V}, a_{\mathbb{Z}(n)}\rangle = 0, \tag{2.6}$$

$$(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}x + (\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)}y + (B_n - z)a_{\mathbb{Z}(n)} = 0. \tag{2.7}$$

Here

$$\langle a, b\rangle \;\equiv\; \langle a, b\rangle_{\mathbb{Z}(n)} = \sum_{k\in\mathbb{Z}(n)} a(k)b(k) \tag{2.8}$$

(no complex conjugation). To solve (2.7) for $a_{\mathbb{Z}(n)}$ we need to analyze the operator

$$z - B_n : l^2(\mathbb{Z}(n)) \to l^2(\mathbb{Z}(n)).$$

Denote by $B_n^I$ the diagonal part of $B_n$,

$$B_n^I(k,j) = \pi^2(k^2 - n^2)\delta_{kj} \qquad (k, j \in \mathbb{Z}(n))$$

and define $B_n^{II} := B_n - B_n^I$. Notice that, with $M \geq 10$, for any $n \geq \frac{M}{2}$ and $|z| \leq M$, $z - B_n^I$ is invertible. Denote by $\| V \|$ the norm of $V$ in $L^2(S^1)$, and introduce

$$T_n := B_n^{II}(z - B_n^I)^{-1} : l^2(\mathbb{Z}(n)) \to l^2(\mathbb{Z}(n)), \tag{2.9}$$

$$n_0 := \max(\frac{M+1}{2}, \| V \|), \quad M \geq 10. \tag{2.10}$$

**Lemma 2.1.** *For $n \geq n_0$ and $|z| \leq M$,*

$$|||T_n|||_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \frac{\| V \|}{5n} \leq \frac{1}{5}; \quad |||(z - B_n^I)^{-1}|||_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \frac{1}{5n}; \tag{2.11}$$

$$z - B_n \textit{ is invertible and } |||(z - B_n)^{-1}|||_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \frac{1}{4n}. \tag{2.12}$$

*Remark.* The conditions in Lemma 2.1 (and subsequent lemmas) are only assumed to insure that the quantities involved are well defined.

*Proof.* To obtain estimates (2.11) notice that $|||T_n|||_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \| T_n \|_{HS}$, where $\| T_n \|_{HS}$ denotes the Hilbert-Schmidt norm of $T_n$:

$$\begin{aligned} \| T_n \|_{HS}^2 &= \sum_{j,k\in\mathbb{Z}(n)} \frac{|\hat{V}(k-j)|^2}{|z - \pi^2(k^2 - n^2)|^2} \leq \| V \|^2 \sum_{k\neq\pm n} \frac{1}{|\pi^2|k^2 - n^2| - M|^2} \\ &\leq \frac{\| V \|^2}{75} \sum_{k\neq\pm n} \frac{1}{|k^2 - n^2|^2} \leq \frac{\pi^2}{300}\; \frac{\| V \|^2}{n^2} \;<\; \frac{\| V \|^2}{30n^2}\;. \end{aligned} \tag{2.13}$$

Similarly,

$$\begin{aligned} |||(z - B_n^I)^{-1}|||^2_{\mathcal{L}(l^2(\mathbb{Z}(n)))} &\leq \| (z - B_n^I)^{-1} \|_{HS}^2 \\ &\leq \sum_{k\neq\pm n} \frac{1}{|\pi^2|k^2 - n^2| - M|^2} \leq \frac{1}{30n^2}. \end{aligned}$$

To prove (2.12), write

$$z - B_n = z - B_n^I - B_n^{II} = (Id_n - B_n^{II}(z - B_n^I)^{-1})(z - B_n^I) = (Id_n - T_n)(z - B_n^I).$$

Then $(z - B_n)^{-1} = (z - B_n^I)^{-1}(Id_n - T_n)^{-1}$ and

$$|||(z - B_n)^{-1}|||_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \le \frac{1}{5n}\sum_{k\ge 0} \| T_n \|^k \le \frac{1}{5n}\frac{1}{1-\frac{1}{5}} = \frac{1}{4n}. \quad \square$$

In view of Lemma 2.1, (2.7) can be solved for $a_{\mathbb{Z}(n)}$, if $n \ge n_0$ and $|z| \le M$:

$$a_{\mathbb{Z}(n)} = (z - B_n)^{-1}(\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)}\ x + (z - B_n)^{-1}(\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)} y.$$

If this is substituted into (2.5) and (2.6), we obtain (with $B_{-n} := B_n$)

$$\begin{pmatrix} -z + \alpha(-n,z) & \hat{V}(-2n) + \beta(-n,z) \\ \hat{V}(2n) + \beta(n,z) & -z + \alpha(n,z) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.14}$$

where

$$\alpha(n,z) := \quad \langle \mathcal{S}^{-n}\mathcal{J}\hat{V}, (z - B_n)^{-1}(\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)}\rangle \tag{2.15}$$

and

$$\beta(n,z) := \quad \langle \mathcal{S}^{-n}\mathcal{J}\hat{V}, (z - B_n)^{-1}(\mathcal{S}^{n}\hat{V})_{\mathbb{Z}(n)}\rangle. \tag{2.16}$$

To analyze (2.14) we begin by investigating $\alpha(n,z)$.

**Lemma 2.2.** *For $n \ge n_0$ and $|z| \le M$,*

$$\alpha(n,z) = \alpha(-n,z)\ .$$

*Proof.* By Lemma 2.3 (iii) below, $(z - B_n)^{-1}(k,j) = (z - B_n)^{-1}(-j,-k)$ and thus

$$\begin{aligned} \alpha(n,z) &= \sum_{k,j\neq\pm n} \hat{V}(n-k)(z - B_n)^{-1}(k,j)\hat{V}(j-n) \\ &= \sum_{k,j\neq\pm n} \hat{V}(-k-(-n))(z - B_n)^{-1}(-j,-k)\hat{V}((-n)-(-j)) \\ &= \sum_{k',j'\neq\pm n} \hat{V}((-n)-k')(z - B_n)^{-1}(k',j')\ \hat{V}(j'-(-n)) \\ &= \alpha(-n,z). \quad \square \end{aligned}$$

**Lemma 2.3.** *For $n \ge n_0$ and $j,k \in \mathbb{Z}(n)$,*

(i) $A_n(k,j) = A_n(-j,-k)$,
(ii) $(A_n - \lambda)^{-1}(k,j) = (A_n - \lambda)^{-1}(-j,-k)$,
(iii) $(z - B_n)^{-1}(k,j) = (z - B_n)^{-1}(-j,-k)$.

*Proof.* (i) Recall that, for $k,j \in \mathbb{Z}(n)$, $A_n(k,j) = k^2\pi^2\delta_{kj} + \hat{V}(k-j)$. Therefore $A_n(k,j) = (-k)^2\pi^2\delta_{(-k)(-j)} + \hat{V}((-j)-(-k)) = A_n(-j,-k)$.

(ii) is a straightforward verification, and (iii) follows from (ii). $\square$

By Lemma 2.2, the system of equations (2.14) with $n \ge n_0$ has a nontrivial solution $\binom{x}{y}$ for some $|z| \le M$, if there exists $z \in \mathbb{C}$, $|z| \le M$, such that

$$(z - \alpha(n,z))^2 - (\hat{V}(-2n) + \beta(-n,z))(\hat{V}(2n) + \beta(n,z)) = 0. \tag{2.17}$$

Equation (2.17) is solved in two steps:

$$z_n = \alpha(n, z_n) + \zeta \quad (\zeta \text{ in } \mathcal{D}_{\frac{M}{2}} := \{\zeta \in \mathbb{C} \mid |\zeta| < \tfrac{M}{2}\}) \tag{2.18}$$

and, with $z(\zeta_n) := z_n(\zeta_n)$ given by (2.18),

$$\zeta_n^2 - \Big(\hat{V}(-2n) + \beta(-n, z(\zeta_n))\Big)\Big(\hat{V}(2n) + \beta(n, z(\zeta_n))\Big) = 0. \tag{2.19}$$

Let us first discuss equation (2.18). To solve it, we use the contractive mapping principle. For that purpose, we need

**Lemma 2.4.** *For $|z| \leq M$ and $n \geq n_0$,*

(i) $|\alpha(n,z)| \leq \frac{1}{4n} \parallel V \parallel^2$,
(ii) $|\frac{d}{dz}\alpha(n,z)| \leq \frac{1}{16n^2} \parallel V \parallel^2$ .

*Proof.* (i) By (2.12) of Lemma 2.1, for $n \geq n_0$ and $|z| \leq M$,

$$|\alpha(n,z)| \leq \parallel V \parallel |||(z - B_n)^{-1}||| \parallel V \parallel \leq \frac{\parallel V \parallel^2}{4n}.$$

(ii) Notice that

$$\frac{d}{dz}\alpha(n,z) = \langle \mathcal{S}^{-n}\mathcal{J}\hat{V}, -(z - B_n)^{-2}\mathcal{S}^{-n}\hat{V}\rangle$$

and therefore

$$|\frac{d}{dz}\alpha(n,z)| \leq \parallel V \parallel |||(z - B_n)^{-1}|||^2 \parallel V \parallel \leq \frac{\parallel V \parallel^2}{16n^2}. \qquad \square$$

Introduce

$$n_1 := \max(n_0, \parallel V \parallel^2) \tag{2.20}$$

**Proposition 2.5.** *For $\zeta \in \mathcal{D}_{M/2}$ and $n \geq n_1$, the equation*

$$z_n = \alpha(n, z_n) + \zeta$$

*has a unique solution $z_n = z_n(\zeta)$ in $\mathcal{D}_M$, which depends analytically on $\zeta \in \mathcal{D}_{\frac{M}{2}}$.*

*Proof.* By Lemma 2.4, since $n \geq n_1$,

$$|\alpha(n,z)| \leq \quad \frac{\parallel V \parallel^2}{4n} \leq \frac{1}{4} \quad (z \in \bar{\mathcal{D}}_M).$$

Therefore, $F(z) \equiv F_{n,\zeta}(z) := \zeta + \alpha(n,z)$ defines a map $F : \bar{\mathcal{D}}_M \to \bar{\mathcal{D}}_M$ (use $|F(z)| \leq |\zeta| + |\alpha(n,z)| \leq \frac{M}{2} + \frac{1}{4} < M$ for $M \geq 10$). $F$ is a contraction, as for any pair $z_1, z_2 \in \bar{\mathcal{D}}_M$

$$|F(z_1) - F(z_2)| \le (\sup_{|z|\le M} |\frac{d}{dz}\alpha(n,z)|)|z_1 - z_2|$$
$$\le \frac{1}{16n}(\frac{1}{n} \parallel V \parallel^2) \cdot |z_1 - z_2| \le \frac{1}{n} \cdot \frac{1}{16} \cdot |z_1 - z_2| \ \le \frac{1}{2} \ |z_1 - z_2|.$$

Thus, for any $\zeta \in \mathcal{D}_{\frac{M}{2}}$ and $n \ge n_1$, $F$ admits a unique fixed point $z_n = z_n(\zeta)$, and $z_n(\zeta)$ depends analytically on $\zeta$. □

It remains to consider (2.19), which requires an estimate of $\beta(\pm n, z)$. First we need some auxilary results. In (2.9) we introduced the operator

$$T_n := B_n^{II}(z - B_n^I)^{-1} \in \mathcal{L}(l^2(\mathbb{Z}(n)).$$

This operator can also be viewed as an element in $\mathcal{L}\Big(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n))\Big)$, where $w = (w(j))_{j\in\mathbb{Z}}$ is the weight $w(j) := (1 + |\frac{j}{2}|)^N \ e^{\frac{\omega}{2}|j|}$ and $(\mathcal{S}^n w)(j) := w(j+n)$. Denote by $W_n : l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)) \to l^2(\mathbb{Z}(n))$ the operator given by

$$W_n(k,j) := w(k+n)\delta_{kj}. \tag{2.21}$$

Notice that $W_n$ is an isometry. Therefore the operator norm of

$$\tilde{T}_n := W_n T_n W_n^{-1} \tag{2.22}$$

is given by

$$|||\tilde{T}_n|||_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \ = \ |||T_n|||_{\mathcal{L}(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)))}. \tag{2.23}$$

**Lemma 2.6.** *For $n \ge n_0$ and $|z| \le M$*

$$|||T_n|||_{\mathcal{L}(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)))} \ \le \ \frac{||V||_{N,\omega}}{5n} \ . \tag{2.24}$$

*Proof.* In view of (2.23), it suffices to estimate the Hilbert Schmidt norm of $\tilde{T}_n$ in $\mathcal{L}(l^2(\mathbb{Z}(n)))$. It follows from the submultiplicative property of the weight $w$ (cf. the definition at the beginning of section 2) that

$$\frac{(\mathcal{S}^n w)(j)}{(\mathcal{S}^n w)(k)} \ \le w(j-k). \tag{2.25}$$

Therefore

$$\begin{aligned}\parallel \tilde{T}_n \parallel_{HS}^2 &= \sum_{j,k\ne\pm n} \frac{|\mathcal{S}^n w(j)|^2}{|\mathcal{S}^n w(k)|^2} \ |\hat{V}(j-k)|^2 \ \frac{1}{|z - \pi^2(k^2-n^2)|^2} \\ &\le \sum_{j,k\ne\pm n} |w(j-k)|^2 |\hat{V}(j-k)|^2 \ \frac{1}{75} \ \frac{1}{(k-n)^2(k+n)^2} \\ &\le \frac{1}{75}\sum_i (|w(i) \ \hat{V}(i)|^2 \sum_{j-i\ne\pm n} \frac{1}{((j-i)^2-n^2)^2}) \\ &\le \parallel V \parallel_{N,\omega}^2 \ \frac{1}{30n^2},\end{aligned}$$

where, for the last inequality, we argue in the same way as in the last steps of the inequality (2.13). □

Let

$$n_2 := \ \max(n_1, \parallel V \parallel_{N,\omega}) = \max(\frac{M+1}{2}, \parallel V \parallel^2, \ \parallel V \parallel_{N,\omega}). \tag{2.26}$$

**Proposition 2.7.** *For* $n \geq n_2$,

(2.27)
$$\Big(\sum_{n\geq n_2}(1+\frac{n}{2})^{2N+2}\ e^{2\omega n}\ \sup_{|z|\leq M}\ |\ \beta(\pm n,z)|^2\Big)^{1/2}\ \leq \|\ V\ \|^2_{N,\omega}\ (1+\frac{1}{8}\ \|\ V\ \|_{N,\omega});$$

(2.28)
$$\Big(\sum_{n\geq n_2}(1+\frac{n}{2})^{2N+4}\ e^{2\omega n}\ \sup_{|z|\leq M}\ |\frac{d}{dz}\beta(\pm n,z)|^2\Big)^{1/2}\ \leq \frac{1}{4}\ \|\ V\ \|^2_{N,\omega}\ (1+\|\ V\ \|_{N,\omega}).$$

*Proof.* The estimates for $\beta(n,z)$ and $\beta(-n,z)$ are obtained in the same way. Let us concentrate on $\beta(n,z)$.

(i) Proof of (2.27): By Lemma 2.6 and (2.26), $(Id_n - T_n) \in \mathcal{L}(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)))$ is invertible for $n \geq n_2$. With

$$(Id_n - T_n)^{-1} = Id_n + T_n(Id_n - T_n)^{-1}$$

and $a_n \equiv a_n(z) \in l^2_w(\mathbb{Z}(n))$ defined by

(2.29)
$$\mathcal{S}^n a_n := (Id_n - T_n)^{-1}\ \mathcal{S}^n\hat{V}\ \in\ l^2_{\mathcal{S}^n w}\ (\mathbb{Z}(n))$$

the expression $\beta(n,z)$ takes the form $\beta(n,z) = \beta_1(n,z) + \beta_2(n,z)$ with

(2.30A)
$$\beta_1(n,z) := \langle \mathcal{S}^{-n}\mathcal{J}\hat{V}, (z - B^I_n)^{-1}\mathcal{S}^n\hat{V}\rangle,$$

(2.30B)
$$\beta_2(n,z) := \langle \mathcal{S}^{-n}\mathcal{J}\hat{V}, (z - B^I_n)^{-1}\ T_n\mathcal{S}^n a_n\rangle.$$

The two terms $\beta_1(n,z)$ and $\beta_2(n,z)$ are estimated separately: For $n \geq n_2$

$$\sup_{|z|\leq M}|\beta_1(n,z)| \leq \sum_{k\neq\pm n}|\hat{V}(n-k)|\ \frac{1}{8|k^2-n^2|}|\ \hat{V}(n+k)| = \frac{1}{8}(b_{\hat{V}} * b_{\hat{V}})(2n),$$

where

$$b_{\hat{V}}(j) := \frac{|\hat{V}(j)|}{j}\ (j\neq 0)\quad \text{and}\quad b_{\hat{V}}(0) := 0.$$

Notice that $\|\ b_{\hat{V}}\ \|_{N+1,\frac{\omega}{2}} \leq 2^{N+1}\ \ \|\ V\ \|_{N,\omega}$ .

Using the fact that $\|\ a * b\ \| \leq (\sum_k |a(k)|)\ \|\ b\ \|$ for $(a(k))_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$ and $(b(k))_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, we conclude that

$$\sum_{n\geq n_2}(1+\frac{n}{2})^{2N+2}\ e^{2\omega n}\ \sup_{|z|\leq M}\ |\beta_1(n,z)|$$
$$\leq \frac{1}{64}\ \sum_{n\neq 0}\ |(b_{\hat{V}} * b_{\hat{V}})(2n)|^2(1+\frac{|n|}{2})^{2N+2}e^{\omega(2|n|)}$$
$$\leq \frac{1}{64}\pi^2(2\ \|\ V\ \|_{N,\omega})^4 = (\frac{\pi}{2}\ \|\ V\ \|^2_{N,\omega})^2\ .$$

Direct computations furnish a slightly better estimate:

$$\begin{aligned}
&\Big(\sum_{n\geq n_2}(1+\frac{n}{2})^{2N+2}\ e^{2\omega n}\ \sup_{|z|\leq M}|\beta_1(n,z)|^2\Big)^{1/2}\\
(2.31)\qquad &\leq \frac{1}{4}\Big(\sum_{j\neq 0}\ \frac{e^{\frac{\omega}{2}|j|}\ |\hat{V}(j)|}{|j|}\ +\ \sum_{j\neq 0}\ \frac{e^{\frac{\omega}{2}|j|}\ |\hat{V}(j)|}{|j|}\Big)\ \parallel V\parallel_{N,\omega}\\
&\leq \frac{1}{2}\Big(\sum_{j\neq 0}\ \frac{1}{j^2}\Big)^{1/2}\ \parallel V\parallel^2_{N,\omega}\leq \parallel V\parallel^2_{N,\omega}\ .
\end{aligned}$$

Next let us estimate $\beta_2(n,z)$ in (2.30B):

$$\begin{aligned}
&(1+\frac{n}{2})^{N+1}\ e^{\omega n}|\beta_2(n,z)|\\
&=(1+\frac{n}{2})^{N+1}\ e^{\omega n}|\langle\mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-1}W_n^{-1}\tilde{T}_nW_n\mathcal{S}^na_n\rangle|\\
&\leq\frac{1}{4}\sum_{\substack{|k-n|\geq n\\ k\neq\pm n}}(1+\frac{|k-n|}{2})^N\ e^{\frac{\omega}{2}|n-k|}|\hat{V}(n-k)|\ \frac{1}{|n+k|}\\
&\quad\times\frac{1}{(1+\frac{|n+k|}{2})^N}|(\tilde{T}_nW_n\mathcal{S}^na_n)(k)|\\
&+\frac{1}{4}\sum_{\substack{|k-n|<n\\ k\neq\pm n}}\frac{e^{\frac{\omega}{2}|n-k|}|\hat{V}(n-k)|}{|n-k|}\ |(\tilde{T}_nW_n\mathcal{S}^na_n)(k)|\\
&\leq\frac{1}{2}\ \parallel V\parallel_{N,\omega}|||\tilde{T}_n|||\ \parallel W_n\mathcal{S}^na_n\parallel.
\end{aligned}$$

By Lemma 2.6 and (2.23), $|||\tilde{T}_n|||\leq\frac{\|V\|_{N,\omega}}{5n}$. Further, $|||(Id_n-T_n)^{-1}|||\leq\frac{5}{4}$ for $n\geq n_2$, and therefore, by (2.29),

$$\parallel W_n\mathcal{S}^na_n\parallel_{l^2(\mathbb{Z}(n))}=\parallel\mathcal{S}^na_n\parallel_{l^2_{\mathcal{S}^nw}(\mathbb{Z}(n))}\leq\frac{5}{4}\parallel V\parallel_{N,\omega}\quad .$$

Combining the estimates above, we obtain

$$\sup_{|z|\leq M}(1+\frac{n}{2})^{N+1}\ e^{\omega n}|\beta_2(n,z)|\leq\frac{\parallel V\parallel^3_{N,\omega}}{8n}\ .$$

Therefore

$$\Big(\sum_{n\geq n_2}\ (1+\frac{n}{2})^{2(N+1)}\ e^{2\omega n}\ \sup_{|z|\leq M}|\beta_2(n,z)|^2\Big)^{1/2}$$

$$\leq\frac{\parallel V\parallel^3_{N,\omega}}{8}\ \Big(\sum_{n\geq n_2}\ \frac{1}{n^2}\Big)^{1/2}\leq\frac{1}{8}\parallel V\parallel^3_{N,\omega}\ .$$

Combined with (2.31), (2.30A) and (2.30B), the estimate (2.27).

(ii) Proof of (2.28): The derivative $\frac{d}{dz}\beta(n,z)$ is given by

(2.32)
$$\begin{aligned}\frac{d}{dz}\beta(n,z) &= \frac{d}{dz}\Big(\langle \mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-1}\mathcal{S}^n\hat{V}\rangle + \langle \mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-1}T_n\mathcal{S}^n a_n\rangle\Big)\\ &= -\langle \mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-2}\mathcal{S}^n\hat{V}\rangle - \langle \mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-2}T_n\mathcal{S}^n a_n\rangle\\ &\quad - \langle \mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-1}(Id_n-T_n)^{-1}T_n(z-B_n^I)^{-1}\mathcal{S}^n a_n\rangle\end{aligned}$$

where we used the fact that the derivative of

$$T_n\mathcal{S}^n a_n = T_n(Id_n-T_n)^{-1}\mathcal{S}^n\hat{V} = ((Id_n-T_n)^{-1}-Id_n)\mathcal{S}^n\hat{V}$$

is given by

$$\frac{d}{dz}T_n\mathcal{S}^n a_n = -(Id_n-T_n)^{-1}T_n(z-B_n^I)^{-1}\mathcal{S}^n a_n.$$

The three terms on the right hand side of (2.32) are estimated separately. The first one is estimated similarly as in (i): Using (2.31), one obtains

(2.32i)
$$\begin{aligned}&\Big(\sum_{n\geq n_2}\sup_{|z|\leq M}(1+\frac{n}{2})^{2N+4}\ e^{2\omega n}|\langle \mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-2}\mathcal{S}^n\hat{V}\rangle|^2\Big)^{1/2}\\ &\quad\leq \frac{1}{4}\parallel V\parallel_{N,\omega}^2 .\end{aligned}$$

The second term on the right hand side of (2.32) is estimated similarly as in (i), and one obtains

(2.32ii)
$$\begin{aligned}&\Big(\sum_{n\geq n_2}\sup_{|z|\leq M}(1+\frac{n}{2})^{2N+4}\ e^{2\omega n}|\langle \mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-2}T_n\mathcal{S}^n a_n\rangle|^2\Big)^{1/2}\\ &\quad\leq \frac{1}{8}\parallel V\parallel_{N,\omega}^3 .\end{aligned}$$

To estimate the last term on the right hand side of (2.32), we first notice that

$$(Id_n-T_n)^{-1}T_n = W_n^{-1}(Id_n-\tilde{T}_n)^{-1}\tilde{T}_nW_n.$$

Thus, with $\tilde{S}_n := (Id_n-\tilde{T}_n)^{-1}\tilde{T}_n$,

$$\begin{aligned}&(1+\frac{n}{2})^{N+2}\ e^{\omega n}|\langle \mathcal{S}^{-n}\mathcal{J}\hat{V},(z-B_n^I)^{-1}(Id_n-T_n)^{-1}T_n(z-B_n^I)^{-1}\mathcal{S}^n a_n\rangle|\\ &\quad\leq \frac{1}{4}\sum_{\substack{|k-n|\geq n\\ k\neq\pm n}}(1+\frac{|k-n|}{2})^N\ e^{\frac{\omega}{2}|n-k|}|\hat{V}(n-k)|\\ &\qquad\times\Big|(\tilde{S}_nW_n(1+n)(z-B_n^I)^{-1}\mathcal{S}^n a_n)(k)\Big|\\ &\qquad+\frac{1}{4}\sum_{\substack{|k-n|<n\\ k\neq\pm n}}\frac{e^{\frac{\omega}{2}|n-k|}|\hat{V}(n-k)|}{|n-k|}\ \Big|(\tilde{S}_nW_n(1+n)(z-B_n^I)^{-1}\mathcal{S}^n a_n)(k)\Big|\\ &\quad\leq \frac{1}{4}\parallel V\parallel_{N,\omega}\parallel \tilde{S}_nW_n(1+n)(z-B_n^I)^{-1}\mathcal{S}^n a_n\parallel\\ &\qquad+\frac{1}{4}\parallel V\parallel_{0,\omega}\parallel \tilde{S}_nW_n(1+n)(z-B_n^I)^{-1}\mathcal{S}^n a_n\parallel\\ &\quad\leq \frac{1}{2}\parallel V\parallel_{N,\omega}|||\tilde{T}_n|||\ |||(Id_n-\tilde{T}_n)^{-1}|||\ \parallel W_n(1+n)(z-B_n^I)^{-1}\mathcal{S}^n a_n\parallel\ .\end{aligned}$$

By Lemma 2.6 and (2.23), $|||\tilde{T}_n||| \leq \frac{\|V\|_{N,\omega}}{5n}$ and $|||(Id_n - \tilde{T}_n)^{-1}||| \leq \frac{5}{4}$. Further, for $n \geq n_2$

$$|||(1+n)(z - B_n^I)^{-1}|||_{\mathcal{L}(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)))} \leq \sup_{\substack{|z|\leq M \\ k\neq \pm n}} \Big| \frac{1+n}{z - \pi^2(k-n)(k+n)} \Big| \leq \frac{1}{4} .$$

Thus, with (2.29),

$$\| W_n(1+n)(z - B_n^I)^{-1} \mathcal{S}^n a_n \|_{l^2(\mathbb{Z}(n))} \leq \frac{1}{4} \cdot \frac{5}{4} \| V \|_{N,\omega} .$$

Combining these estimates leads to

(2.32iii)

$$\Big( \sum_{n \geq n_2} \sup_{|z|\leq M} (1 + \frac{n}{2})^{2N+4} \times e^{2\omega n} |\langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1}(Id_n - T_n)^{-1} T_n (z - B_n^I)^{-1} \mathcal{S}^n a_n \rangle|^2 \Big)^{1/2}$$
$$\leq \frac{1}{32} \| V \|^3_{N,\omega} ( \sum_{n\geq n_2} \frac{1}{n^2})^{1/2} \leq \frac{1}{32} \| V \|^3_{N,\omega} .$$

From (2.32) and (2.32i)-(2.32iii) the estimate (2.28) follows. □

We are now ready to investigate (2.19).
Let

(2.33) $$r_n := \max(|\hat{V}(\pm 2n)|) + \max_{|z|\leq M} |\beta(\pm n, z)|.$$

Notice that, by Proposition 2.7, for $n \geq n_2$,

(2.34) $$r_n \leq \| V \| + \| V \|^2_{N,\omega} (1 + \frac{1}{8} \| V \|_{N,\omega}).$$

**Proposition 2.8.** *Assume that $M \geq 10$ satisfies*

$$\| V \| + \| V \|^2_{N,\omega} (1 + \frac{1}{8} \| V \|_{N,\omega}) \leq \frac{M}{4} .$$

*Then, for $n \geq n_2$, equation* (2.19) *has exactly two (counted with multiplicity) solutions $\zeta_n^+, \zeta_n^-$ in $\overline{\mathcal{D}_{r_n}}$.*

*Proof.* The result follows from Rouché's theorem. Clearly $\zeta^2 = 0$ has two roots in $\mathcal{D}_{r_n}$. For $|\zeta| = Kr_n$ with $1 < K < 2$,

$$\sup_{|z|\leq M} \Big| (\hat{V}(2n) + \beta(n,z))(\hat{V}(-2n) + \beta(-n,z)) \Big| \leq r_n^2 < |\zeta^2|.$$

As $\beta(\pm n, z_n(\zeta))$ depend analytically on $\zeta$ for $|\zeta| < \frac{M}{2}$ and $Kr_n < 2r_n \leq \frac{M}{2}$, we deduce from Rouché's theorem that equation (2.19) has precisely two roots in $\mathcal{D}_{Kr_n}$. As the two roots are independent of $K$ and $1 < K < 2$ is arbitrary close to 1, we conclude that $\zeta_n^\pm \in \overline{\mathcal{D}_{r_n}}$. □

Let $z_n^\pm = z(\zeta_n^\pm) = \zeta_n^\pm + \alpha(n, z_n^\pm)$, where $\zeta_n^\pm$ are given by Proposition 2.8. Then

(2.35) $$|z_n^+ - z_n^-| \leq |\zeta_n^+ - \zeta_n^-| + ( \sup_{|z|\leq M} | \frac{d}{dz} \alpha(n,z)|) |z_n^+ - z_n^-|.$$

By Lemma 2.4 (ii), (2.20) and as $n_2 \geq n_1$,

$$\sup_{|z|\leq M} |\frac{d}{dz}\alpha(n,z)| \leq \frac{1}{2}. \tag{2.36}$$

Together with $|\zeta_n^+ - \zeta_n^-| \leq |\zeta_n^+| + |\zeta_n^-| \leq 2r_n$, estimate (2.35) leads to

$$|z_n^+ - z_n^-| \leq 4r_n.$$

In view of the definition (2.33) of $r_n$, Proposition 2.7 and $|z_n^+ - z_n^-| = |\lambda_n^+ - \lambda_n^-|$, we conclude that

$$\begin{aligned}(&\sum_{n\geq n_2} (1+\frac{n}{2})^{2N} e^{2\omega n} |\lambda_n^+ - \lambda_n^-|^2)^{1/2} \\ &\leq 8(\| V \|_{N,\omega} + \| V \|_{N,\omega}^2 (1 + \frac{1}{8} \| V \|_{N,\omega})) < \infty.\end{aligned} \tag{2.37}$$

Next we want to obtain asymptotics for $\lambda_n^+ - \lambda_n^-$. Rewrite (2.19) in two ways:

$$(\zeta_n^{\pm})^2 - \delta(n)\delta(-n) - \eta_n^{\pm} = 0, \tag{2.38}$$

with $\eta_n^{\pm} = \eta(z_n^{\pm})$, where $\delta(n), \eta(z)$ are defined either by (alternative 1)

$$\begin{aligned}\delta(n) \equiv \delta^I(n) &:= \hat{V}(2n) + \beta(n,0),\\ \eta(z) \equiv \eta^I(z) &:= (\beta(-n,z) - \beta(-n,0))\hat{V}(2n)\\ &\quad + (\beta(n,z) - \beta(n,0))\hat{V}(-2n)\\ &\quad + \beta(-n,z)\beta(n,z) - \beta(-n,0)\beta(n,0)\end{aligned}$$

or by (alternative 2)

$$\begin{aligned}\delta(n) \equiv \delta^{II}(n) &:= \hat{V}(2n),\\ \eta(z) \equiv \eta^{II}(z) &:= \beta(-n,z)\hat{V}(2n) + \beta(n,z)\hat{V}(-2n) + \beta(-n,z)\beta(n,z).\end{aligned}$$

Introduce $s_n$, given by (alternative 1)

$$\begin{aligned}s_n \equiv s_n^I := \sup_{|z|\leq M} \Big(&|\beta(-n,z) - \beta(-n,0)||\hat{V}(2n)| + |\beta(n,z) - \beta(n,0)||\hat{V}(-2n)|\\ &+ |\beta(-n,z)||\beta(n,z)| + |\beta(-n,0)||\beta(n,0)|\Big)\end{aligned}$$

or by (alternative 2)

$$s_n \equiv s_n^{II} := \sup_{|z|\leq M} \Big(|\beta(-n,z)||\hat{V}(2n)| + |\beta(n,z)||\hat{V}(-2n)| + |\beta(-n,z)||\beta(n,z)|\Big).$$

Use $|\beta(n,z) - \beta(n,0)| \leq M \sup_{|z|\leq M} |\frac{d}{dz}\beta(n,z)|$ and Proposition 2.7 to obtain

$$\begin{aligned}&\sum_{n\geq n_2} (1+\frac{n}{2})^{2N+2} e^{2\omega n} s_n^I\\ &\leq \frac{M}{2} \| V \|_{N,\omega}^3 (1 + \| V \|_{N,\omega}) + 2 \| V \|_{N,\omega}^4 (1 + \frac{1}{8} \| V \|_{N,\omega})^2\end{aligned} \tag{2.39$^{\mathrm{I}}$}$$

and

$$\begin{aligned}&\sum_{n\geq n_2} (1+\frac{n}{2})^{2N+1} e^{2\omega n} s_n^{II}\\ &\leq 2 \| V \|_{N,\omega}^3 (1 + \frac{1}{8} \| V \|_{N,\omega}) + \| V \|_{N,\omega}^4 (1 + \frac{1}{8} \| V \|_{N,\omega})^2 .\end{aligned} \tag{2.39$^{\mathrm{II}}$}$$

**Proposition 2.9.** *Assume that $M \geq 10$ satisfies*

$$\| V \| + \| V \|^2_{N,\omega} (1 + \frac{1}{8} \| V \|_{N,\omega}) \leq \frac{M}{4}.$$

*Then for $n \geq n_2$ and $\delta(n), s_n$ given by either $\delta^I(n), s_n^I$ or $\delta^{II}(n), s_n^{II}$, the roots $\zeta_n^+, \zeta_n^-$ can be labeled in such a way that*

(i) $|\zeta_n^+ - (\delta(n)\delta(-n))^{1/2})| \leq 5 s_n^{1/2}$,

(ii) $|\zeta_n^- + (\delta(n)\delta(-n))^{1/2}| \leq 5 s_n^{1/2}$ .

*Proof.* We consider two different cases:

*Case* 1. $|\delta(n)\delta(-n)| \leq 4 s_n$. This is an easy case for which (i) and (ii) are proved in the same way. Let us concentrate on (i). Then

$$\begin{aligned} |(\zeta_n^+ - (\delta(n)\delta(-n))^{1/2})^2| \ &\leq \ 2|(\zeta_n^+)^2| + 2|\delta(n)\delta(-n)| \\ &\leq 2|\delta(n)\delta(-n) + \eta_n^+| + 2|\delta(n)\delta(-n)| \\ &\leq 4|\delta(n)\delta(-n)| + 2|\eta_n^+| \ \leq 18 s_n \leq (5 s_n^{1/2})^2, \end{aligned}$$

where for the second inequality, we used (2.38).

Thus $|\zeta_n^+ - (\delta(n)\delta(-n))^{1/2}| \ \leq 5 s_n^{1/2}$.

*Case* 2. $|\delta(n)\delta(-n)| \geq 4 s_n$. Without any loss of generality we may assume that $s_n > 0$. In particular, $|\delta(n)\delta(-n)| > 0$. The equation (2.38) can than be rewritten as

$$(\zeta_n)^2 = \delta(-n)\delta(n)\Big(1 + \frac{\eta(z(\zeta_n))}{\delta(-n)\delta(n)}\Big) \tag{2.40}$$

where $z(\zeta_n)$ is given by (2.18). With

$$\xi := \frac{\zeta_n}{(\delta(-n)\delta(n))^{1/2}}$$

formula (2.40) leads to

$$(\xi)^2 = \Big(1 + \frac{\eta(z(\zeta_n))}{(\delta(-n)\delta(n))}\Big) \ . \tag{2.41}$$

As $|\delta(n)\delta(-n)| \geq 4 s_n$, one concludes that $\left| \frac{\eta(z(\zeta_n))}{\delta(-n)\delta(n)} \right| \leq \frac{1}{4}$ .

Denoting by $(1+w)^{1/2}$ the branch of the square root determined by $(1)^{1/2} = +1$, we obtain the equations

$$\xi = \pm F(\xi) = \pm\Big(1 + \frac{\eta(z)}{\delta(n)\delta(-n)}\Big)^{1/2} \tag{2.42$^\pm$}$$

with $z = z(\zeta_n) = z((\delta(n)\delta(-n))^{1/2} \cdot \xi)$ . Let us first consider (2.42$^+$): Introduce $\mathcal{D}_{1/4}(1) := \{\xi \in \mathbb{C} \Big| |\xi - 1| < \frac{1}{4}\}$ and notice that for $\xi \in \mathcal{D}_{1/4}(1), \zeta_n := (\hat{V}(2n)\hat{V}(-2n))^{1/2}\xi$ satisfies $|\zeta_n| \leq \frac{M}{4} \cdot \frac{5}{4} < \frac{M}{2}$.

As $|(1+x)^{1/2} - 1| \leq |x|$ for $x \in \mathcal{D}_{1/4}(0)$ and case 2 holds, we conclude that $F$ maps $\overline{\mathcal{D}_{1/4}(1)}$ into itself. Moreover $F$ is continuous and therefore, according to Brower's fixed point theorem, admits at least one fixed point, denoted by $\xi^I$, i.e.

$$\xi^I = +F(\xi^I) = +\Big(1 + \frac{\eta(z^I)}{\delta(n)\delta(-n)}\Big)^{1/2}$$

where $z^I = z(\Big(\hat V(-2n)\hat V(2n)\Big)^{1/2}\xi^I)$. Then, as we are in case 2,

$$|\xi^I - 1| \le \Big|\Big(1+\frac{\eta(z^I)}{\delta(n)\delta(-n)}\Big)^{1/2} - 1\Big| \le \frac{|\eta(z^I)|}{|\delta(n)\delta(-n)|} \le \frac{1}{2}\,\frac{s_n^{1/2}}{|\delta(n)\delta(-n)|^{1/2}}$$

and, with $\zeta^I := (\delta(n)\delta(-n))^{1/2}\xi^I$,

$$|\zeta^I - (\delta(-n)\delta(n))^{1/2}| \le \frac{1}{2}\, s_n^{1/2}. \tag{2.43}$$

The same arguments can be used to show that there exists a solution $\xi^{II} \in \mathcal{D}_{1/4}(-1)$ of (2.42$^-$) so that, with $\zeta^{II} := (\delta(n)\delta(-n))^{1/2}\,\xi^{II}$,

$$|\zeta^{II} + (\delta(n)\delta(-n))^{1/2}| \le \frac{1}{2}\, s_n^{1/2}\ . \tag{2.44}$$

It remains to show that $\{\zeta^I,\zeta^{II}\} = \{\zeta_n^+,\zeta_n^-\}$. First notice that $\zeta^I \ne \zeta^{II}$. Otherwise, we obtain a contradiction, by combining (2.43), (2.44), $s_n > 0$ and the inequality case 2 as follows: Assume $\zeta^I = \zeta^{II} = \zeta^*$. Then

$$0 < 2(2\, s_n^{1/2}) \le |2(\delta(-n)\delta(n))^{1/2}| \le |(\delta(n)\delta(-n))^{1/2} - \zeta^* + (\delta(n)\delta(-n))^{1/2} + \zeta^*| \le \frac{1}{2}\, s_n^{1/2} + \frac{1}{2}\, s_n^{1/2} = s_n^{1/2},$$

and $0 < 4s_n^{1/2} \le s_n^{1/2}$ gives the claimed contradiction.

Further notice that $\zeta^I, \zeta^{II}, \zeta_n^+, \zeta_n^-$ are all solutions of (2.40). But according to Proposition 2.8, equation (2.40) has precisely two solutions. Therefore $\{\zeta^I,\zeta^{II}\} = \{\zeta_n^+,\zeta_n^-\}$. This proves Proposition 2.9 in case 2. □

We are now ready to prove Theorem 2. It is contained in the following

**Theorem 2.10.** *Let $S^1 = \mathbb{R}/\mathbb{Z}$, $N \in \mathbb{Z}_{\ge 0}$, $\omega \in \mathbb{R}_{\ge 0}$ and $M \ge 10$.*

*Then, for any $V \in H^{N,\omega}(S^1;\mathbb{C})$ with $\| V \| + \| V \|_{N,\omega}^2 (1 + \frac{1}{8} \| V \|_{N,\omega}) \le \frac{M}{4}$,*

(i) $$\Big(\sum_{n\ge M^2} (1+\frac{n}{2})^{2N+1}\, e^{2\omega n}\Big|(\lambda_n^+ - \lambda_n^-) - 2(\hat V(-2n)\hat V(2n))^{1/2}\Big|^2\Big)^{1/2} \le 3M^2,$$

(ii)
$$\Big(\sum_{n\ge M^2} (1+\frac{n}{2})^{2N+2}\, e^{2\omega n} \times \Big|(\lambda_n^+ - \lambda_n^-) - 2((\hat V(-2n) + \beta(-n,0))(\hat V(2n)+\beta(n,0)))^{1/2}\ \Big|^2\Big)^{1/2} \le 5M^2,$$

*where, for $n \ge M^2$, $\lambda_n^\pm = n^2\pi^2 + z_n^\pm$ (and thus $\{\lambda_n^+,\lambda_n^-\} = \{\lambda_{2n},\lambda_{2n-1}\}$) with $(\lambda_n)_{n\ge 0}$ denoting the periodic spectrum (ordered as explained in the introduction) of $-\frac{d^2}{dx^2} + V$ considered on the interval $[0,2]$) and $(\hat V(k))_{k\in\mathbb{Z}}$ are the Fourier coefficients of $V$ when considered as functions with period 2.*

*Proof.* Without any loss of generality we assume that $V \in H_0^{N,\omega}(S^1;\mathbb{C})$. Statements (i) and (ii) are proved in the same way, so we concentrate on (i). Notice that $M^2 \ge n_2 := \max(\frac{M+1}{2}, \| V \|^2, \| V \|_{N,\omega})$.

For $n \geq M^2$, $\lambda_n^+ - \lambda_n^- = z_n^+ - z_n^-$. Furthermore, $z_n^\pm = \alpha(n, z_n^\pm) + \zeta_n^\pm$, and, by Proposition 2.9,

$$|(\zeta_n^+ - \zeta_n^-) - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2}| \leq 10 s_n^{1/2} .$$

Therefore

$$\begin{aligned} &|\lambda_n^+ - \lambda_n^- - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2}| \\ &\leq |\zeta_n^+ - \zeta_n^- - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2}| + \Big(\sup_{|z|\leq M} |\frac{d}{dz}\alpha(n,z)|\Big)|z_n^+ - z_n^-| \\ &\leq 10 s_n^{1/2} + \frac{\| V \|^2}{n^2} |z_n^+ - z_n^-|, \end{aligned} \tag{2.45}$$

where for the last inequality, we used Lemma 2.4 (ii) and (2.26). By (2.37)

$$\begin{aligned} &\Big(\sum_{n\geq M^2} (1+\frac{n}{2})^{2N+4} e^{2\omega n}(\frac{\| V \|^2}{n^2} |z_n^+ - z_n^-|)^2\Big)^{1/2} \\ &\leq 2 \cdot \| V \|^2 8(\| V \|_{N,\omega} + \| V \|_{N,\omega}^2 (1 + \frac{1}{8} \| V \|_{N,\omega})) \leq 2M^2 . \end{aligned} \tag{2.46}$$

By ($2.39^{\mathrm{II}}$),

$$10 \cdot \Big(\sum_{n\geq M^2} (1+\frac{n}{2})^{2N+1} e^{2\omega n} s_n\Big)^{1/2} \leq 5M . \tag{2.47}$$

As $M \geq 10$, $5M \leq M^2$, and therefore from (2.45), (2.46) and (2.47) we obtain

(2.48)

$$\Big(\sum_{n\geq M^2} (1+\frac{n}{2})^{2N+1} e^{2\omega n} \Big| \lambda_n^+ - \lambda_n^- - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2}\Big|^2 \Big)^{1/2} \leq 3M^2 . \qquad \square$$

*Remark* 1. Theorem 2.10 can be improved for real valued potentials. It leads to a result obtained by Marčenko [Ma]:

**Theorem 2.10 A.** *Let* $N \in \mathbb{Z}_{\geq 0}$ *and* $M \geq 10$. *Then, for any* $V \in H^{N,\omega}(S^1;\mathbb{R})$ *with* $\| V \| + \| V \|_{N,\omega}^2 (1 + \frac{1}{8} \| V \|_{N,\omega}) \leq \frac{M}{4}$,

$$\sum_{n\geq M^2} (1+\frac{n}{2})^{2N+2} e^{2\omega n} \Big| (\lambda_n^+ - \lambda_n^-) - 2 \mid \hat{V}(-2n)\hat{V}(2n) \mid^{1/2}\Big|^2 \leq 3M^2 . \tag{2.49}$$

*Remark for Theorem 2.10 A.* Recall that

$$(\zeta_n)^2 - \hat{V}(2n)\hat{V}(-2n) - \eta_n = 0. \tag{2.19}$$

If $V$ is real valued, then $|\hat{V}(2n)| = |\hat{V}(-2n)|$ . This is a quantitative version of the following statement:

$$\hat{V}(2n) \cdot \hat{V}(-2n) = 0 \iff \hat{V}(2n) = 0 \textbf{ and } \hat{V}(-2n) = 0 . \tag{2.50}$$

To improve on Theorem 2.10 as in Theorem 2.10 A, it seems that one needs to restrict to potentials satisfying a vanishing condition which is a quantitative version of (2.50).

*Proof of Theorem 2.10 A.* Without any loss of generality, $V \in H_0^{N,\omega}(S^1;\mathbb{R})$, i.e. $\hat{V}(0) = 0$. If $V$ is real valued, then $\hat{V}(-k) = \overline{\hat{V}(k)}$, and

$$\beta(-n,z) = \overline{\beta(n,\bar{z})} \quad (|z| \le M,\ n \ge n_2), \tag{2.51}$$

$$\overline{\alpha(n,z)} \ = \ \alpha(n,\bar{z}) \quad (|z| \le M,\ n \ge n_2). \tag{2.52}$$

Further, $-\frac{d^2}{dx^2} + V$ is selfadjoint, and therefore, the perodic spectrum of $-\frac{d^2}{dx^2} + V$ is contained in $\mathbb{R}$. Following the proof of Theorem 2.10, we know that for $n \ge 2M^2$ we have $\lambda_n^\pm = n^2\pi^2 + z_n^\pm$, and thus $z_n^\pm \in \mathbb{R}$. Moreover, $\zeta_n^\pm = z_n^\pm - \alpha(n, z_n^\pm) \in \mathbb{R}$, as $\alpha(n, z_n^\pm) \in \mathbb{R}$ by (2.52).

Therefore, equation (2.19) can be written as

$$(\zeta_n^\pm)^2 = |\hat{V}(2n) + \beta(n, z(\zeta_n^\pm))|^2,$$

which leads to

$$\zeta_n^\pm = \pm\ |\hat{V}(2n) + \beta(n, z(\zeta_n^\pm))|\ .$$

Thus

$$\begin{aligned} |\zeta_n^\pm - (\pm|\hat{V}(2n)|)\ | &\le |\ |\hat{V}(2n) + \beta(n, z(\zeta_n^\pm))| - |\hat{V}(2n)|\ | \\ &\le |\beta(n, z(\zeta_n^\pm))| \end{aligned}$$

and, by Proposition 2.7,

$$\begin{aligned} &\Big( \sum_{n \ge n_2} (1 + \frac{n}{2})^{2N+2}\ e^{2\omega n} \ \Big|\ (\zeta_n^+ - \zeta_n^-) - 2|\hat{V}(2n)|\ \Big| \Big)^{1/2} \\ &\quad \le \|\ V\ \|_{N,\omega}^2\ (1 + \frac{1}{8}\ \|\ V\ \|_{N,\omega}) \le \frac{M}{4}\ . \end{aligned} \tag{2.53}$$

Combining with (2.45) and (2.46), one obtains

$$\begin{aligned} &\Big( \sum_{n \ge M^2} (1 + \frac{n}{2})^{2N+2}\ e^{2\omega n} \ \Big|\ (\lambda_n^+ - \lambda_n^-)\ -\ 2|\hat{V}(2n)|\ \Big|^2 \Big)^{1/2} \\ &\quad \le\ 2M^2\ +\ \frac{M}{4}\ \le 3M^2\ . \qquad \square \end{aligned}$$

*Remark* 2. If $V$ is an even, possibly complex valued potential, Theorem 2.10 can be reformulated in a way which leads to an improvement. In the case when $V \in H^{N,\omega}(S^1;\mathbb{C})$ is even, i.e. $V(x) = V(-x)$, it is a well known fact that for $n$ sufficiently large, $\{\lambda_n^+, \lambda_n^-\} = \{\mu_n, \nu_n\}$, where $(\mu_n)_{n\ge1}$ denote the Dirichlet eigenvalues and $(\nu_n)_{n\ge0}$ denote the Neumann eigenvalues of $-\frac{d^2}{dx^2} + V$ considered on the unit interval. Further, the eigenfunctions of the Dirichlet eigenvalues are odd, whereas the eigenfunctions of the Neumann eigenvalues are even.

**Theorem 2.10 B.** *Let $N \in \mathbb{Z}_{\ge0}$, $\omega \in \mathbb{R}_{\ge0}$ and $M \ge 10$.*

*Then, for any* **even** *potential $V \in H^{N,\omega}(S^1, \mathbb{C})$ satisfying*

$$\|\ V\ \| + \|\ V\ \|_{N,\omega}^2\ (1 + \frac{1}{8}\ \|\ V\ \|_{N,\omega})\ \le\ \frac{M}{4},$$

*we have*

$$\Big( \sum_{n \ge M^2} (1 + \frac{n}{2})^{2n+2}\ e^{2\omega n}\ |\ (\mu_n - \nu_n) + 2\hat{V}(2n)|^2 \Big)^{1/2}\ \le\ 3M^2\ .$$

*Proof.* In the case where $V$ is even one verifies that

$$\hat{V}(2k) = \hat{V}(-2k);\ \beta(-k,z) = \beta(k,z)\ .$$

Therefore, equation (2.19) leads to $\zeta_n^2 = (\hat{V}(2n) + \beta(n, z(\zeta_n)))^2$ and, in turn,

$$\zeta_n^{\pm} = \pm\ (\hat{V}(2n) + \beta(n, z(\zeta_n^{\pm}))). \tag{2.53$^\pm$}$$

Then, with $\varepsilon_n = \pm$,

$$\mu_n = n^2\pi^2 + \alpha(n, z(\zeta_n^{\varepsilon_n})) + \varepsilon_n(\hat{V}(2n) + \beta(n, z(\zeta_n^{\varepsilon_n})))$$

and

$$\nu_n = n^2\pi^2\ +\ \alpha(n, z(\zeta_n^{-\varepsilon_n})) - \varepsilon_n(\hat{V}(2n) + \beta(n, z(\zeta_n^{-\varepsilon_n}))).$$

To determine the sign $\varepsilon_n$, recall that the eigenfunction $y_2(x, \mu_n)$ is odd. Its Fourier coefficients $(a(k;n))_{k\in\mathbb{Z}}$ therefore satisfy $a(-k;n) = -a(k;n)$.

Thus, in equation (2.14), $x = -y$. Together with

$$-z(\zeta^{\varepsilon_n}) + \alpha(n, z(\zeta^{\varepsilon_n}))\ = \zeta^{\varepsilon_n}\ =\ \varepsilon_n(\hat{V}(2n) + \beta(n, z(\zeta_n^{\varepsilon_n}))),$$

equation (2.14) implies

$$(\varepsilon_n(\hat{V}(2n) + \beta(n, z(\zeta^{\varepsilon_n})))\ +\ \hat{V}(2n) + \beta(n, z(\zeta^{\varepsilon_n})))x = 0\ .$$

Since, in view of (2.5)-(2.7), $(x,y) \neq (0,0)$ (for $n \geq n_0$) (otherwise, $a(k,n) = 0$ for all $k$) we then conclude that $\varepsilon_n = -1$ as claimed. As a consequence,

$$\mu_n - \nu_n = -2\hat{V}(2n) - \beta(n, z_n^+) - \beta(n, z_n^-) + \alpha(n, z_n^-) - \alpha(n, z_n^+)\ .$$

As in the proof of Theorem 2.10 A, one then obtains, by Proposition 2.7, combined with (2.45) and (2.46),

$$\Big(\sum_{n\geq 2M^2} (1+\frac{n}{2})^{2N+2}\ e^{2\omega n}|\mu_n - \nu_n + 2\hat{V}(2n)|^2\Big)^{1/2} \leq 3M^2\ . \quad \square$$

As an application of Theorem 2.10 we obtain asymptotic estimates of the eigenvalues $\lambda_n^{\pm}$,

$$\begin{aligned}\lambda_n^{\pm} =& n^2\pi^2 + \alpha(n, \frac{z_n^+ + z_n^-}{2}) \\ &\pm ((\hat{V}(-2n) + \beta(-n,0))(\hat{V}(2n) + \beta(n,0)))^{1/2} + l^2_{N+1,\omega}(n),\end{aligned} \tag{2.54$^\pm$}$$

and of $\tau_n := \frac{\lambda_n^+ + \lambda_n^-}{2}$,

$$\tau_n = n^2\pi^2 + \alpha(n, \frac{z_n^+ + z_n^-}{2})\ +\ l^2_{N+1,\omega}(n), \tag{2.55}$$

where, by abuse of notation, we mean by $(l^2_{N+1,\omega}(n))_{n\geq 1}$ an element in $l^2_{N+1,\omega}(\mathbb{N})$.

We finish this section with a brief discussion of the linear space $E_n$ spanned by eigenfunctions $f_n^+$, $f_n^-$ corresponding to simple eigenvalues $\lambda_n^+ \neq \lambda_n^-$, and of the root space $E_n$ corresponding to double eigenvalues $\lambda_n^+ = \lambda_n^-$.

For a function $f(x) = \sum_{k\in\mathbb{Z}}\ \hat{f}(k)\ e^{ik\pi x}$ in $E_n$, the part $\sum_{k\neq\pm n} \hat{f}(k)\ e^{ik\pi x}$ is small when compared with $\hat{f}(n)\ e^{in\pi x}\ +\ \hat{f}(-n)\ e^{-in\pi x}$. This follows from the following result, which we will use in section 3.

In view of (2.7), we introduce $a_{x,y,z}\ \in\ l^2(\mathbb{Z}(n))$:

$$a_{x,y,z} := x(z - B_n)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}\ +\ y(z - B_n)^{-1}(\mathcal{S}^{-n}\hat{V})_{\mathbb{Z}(n)}\ . \tag{2.56}$$

**Proposition 2.11.** *Assume* $V \in H_0^{N,\omega}(S^1;\mathbb{C})$. *Then for* $|z| \leq M,\ n \geq n_2$ *and* $x, y \in \mathbb{C}$

(i) $\| (z - B_n)^{-1}(\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l^2_{\mathcal{S}^n w}(\mathbb{Z}(n))} \leq \frac{1}{n} \| V \|_{N,\omega}$ ;

(ii) $\| (z - B_n)^{-1}(\mathcal{S}^{-n} V)_{\mathbb{Z}(n)} \|_{l^2_{\mathcal{S}^{-n} w}(\mathbb{Z}(n))} \leq \frac{1}{n} \| V \|_{N,\omega}$ ;

(iii) $\left( \sum_{k \neq \pm n} e^{| \ |k|-n \ |\omega} \ (1 + | \frac{|k|-n}{2} |)^{2N} \ |a_{x,y,z}(k)|^2 \right)^{1/2} \leq \frac{|x|+|y|}{n} \| V \|_{N,\omega}$ ;

(iv) $\| (z - B_n)^{-1}(\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l^1(\mathbb{Z}(n))} \leq \frac{1}{n} \| V \|$ ;

(v) $\| (z - B_n)^{-1}(\mathcal{S}^{-n} V)_{\mathbb{Z}(n)} \|_{l^1(\mathbb{Z}(n))} \leq \frac{1}{n} \| V \|$ ;

(vi) $\sum_{k \neq \pm n} |a_{x,y,z}(k)| \leq \frac{|x|+|y|}{n} \| V \|$ .

*Remark.* Notice that, with $w_1 := \mathcal{S}^n w, w_2 := \mathcal{S}^{-n} w$, the function

$$w_1 \wedge w_2(k) := \min(w_1(k), w_2(k))$$

is given by

$$\begin{aligned} w_1 \wedge w_2(k) &= (1 + | \frac{|k|-n}{2} |)^N \ e^{\frac{\omega}{2} | \ |k|-n \ |} \\ &= \begin{cases} w_1(k) & \text{for} \quad k \leq 0, \\ w_2(k) & \text{for} \quad k \geq 0 \ . \end{cases} \end{aligned}$$

Furthermore,

$$\sup_k \frac{w_1(k)}{w_2(k)} \leq (1+n)^N \ e^{n\omega} \quad \text{and} \quad \sup_k \frac{w_2(k)}{w_1(k)} \leq (1+n)^N \ e^{n\omega} \ .$$

This implies that $w_1 \wedge w_2$ is a weight with $M_{w_1 \wedge w_2} = ((1+n)^N \ e^{n\omega})^2$. Thus $M_{w_1 \wedge w_2}$ is increasing in $n$.

*Proof.* (i) By Lemma 2.6,

$$|||T_n|||_{\mathcal{L}(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)))} \leq \frac{\| V \|_{N,\omega}}{2n}$$

and thus, for $|z| \leq M$ and $n \geq n_2$,

$$\begin{aligned} |||(z - B_n)^{-1}|||_{\mathcal{L}(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)))} &= |||(z - B_n^I)^{-1} \cdot (Id_n - T_n)^{-1}|||_{\mathcal{L}(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)))} \\ &\leq 2|||(z - B_n^I)^{-1}|||_{\mathcal{L}(l^2_{\mathcal{S}^n w}(\mathbb{Z}(n)))} \leq 2 \cdot \frac{1}{2n}, \end{aligned}$$

where for the last inequality we used Lemma 2.1. This implies that

$$\| (z - B_n)^{-1}(\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l^2_{\mathcal{S}^n w}(\mathbb{Z}(n))} \leq \frac{1}{n} \ \| \mathcal{S}^n V \|_{\mathcal{S}^n w} \leq \frac{1}{n} \| V \|_{N,\omega} \ .$$

(ii) Using the same arguments as in the proof of Lemma 2.6, one shows that

$$|||T_n|||_{\mathcal{L}(l^2_{\mathcal{S}^{-n} w}(\mathbb{Z}(n)))} \leq \frac{\| V \|_{N,\omega}}{2n} \ .$$

One then argues as in the proof of (i) to conclude (ii).

(iii) Notice that for $k \in \mathbb{Z},\ n \geq 1$

$$| \ |k| - n \ | = \min(|k-n|, \ |k+n \ |) \ .$$

Thus

$$(1 + | \frac{|k|-n}{2} |)^N \ e^{\frac{\omega}{2} | \ |k|-n \ |} \leq \min((\mathcal{S}^n w)(k), \ (\mathcal{S}^{-n} w)(k))$$

and (iii) follows from (i), (ii) and (2.56).

(iv) We have

$$\begin{aligned}\| (z - B_n)^{-1}(\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l^1(\mathbb{Z}(n))} &= \| (z - B_n^I)^{-1}(Id_n - T_n)^{-1}(\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l^1(\mathbb{Z}(n))} \\ &= \sum_{k \neq \pm n} \frac{1}{|z - \pi^2(k^2 - n^2)|} \, |(Id_n - T_n)^{-1}(\mathcal{S}^n V)(k)| \\ &\leq \Big( \sum_{k \neq \pm n} \frac{1}{|z - \pi^2(k^2 - n^2)|^2} \Big)^{1/2} \, \| (Id_n - T_n)^{-1}(\mathcal{S}^n V)_{\mathbb{Z}(n)} \| \\ &\leq \frac{1}{2n} \cdot 2 \cdot \| V \|,\end{aligned}$$

where for the last inequality we used Lemma 2.1 (and its proof).

(v) is proved in the same way as (iv).

(vi) follows from (iv) and (v). □

## 3. Proof of Theorem 1

To prove Theorem 1, we follow the same scheme used in [BKM1, section 2]. Recall that the map $\Lambda$ (cf. [BBGK]) is constructed in two steps. First let us consider the map $\Phi : L_0^2(S^1) \to l^2(\mathbb{R}^2)$, introduced and analyzed in [Ka]. For $V \in L_0^2(S^1)$, denote by $E_n$ the image of the Riesz projector ($n \geq 1$)

$$P_n := \frac{1}{2\pi i} \int_{\Gamma_n} (z - (-\frac{d^2}{dx^2} + V))^{-1} \, dz, \tag{3.1}$$

where $\Gamma_n$ is a counterclockwise oriented circle with center $\tau_n(V) := (\lambda_n^+ + \lambda_n^-)/2$ of radius bigger than $\frac{\gamma_n}{2} = \frac{\lambda_n^+ - \lambda_n^-}{2}$, but sufficiently small so that all eigenvalues different from $\lambda_n^+$, $\lambda_n^-$ are outside of $\Gamma_n$. Here, for convenience, we set $\lambda_n^+ \equiv \lambda_{2n}$ and $\lambda_n^- \equiv \lambda_{2n-1}$.

We choose in $E_n$ a basis

$$\begin{aligned} G_{2n-1}(x) &\equiv G_n^-(x) = \sum_{k \in \mathbb{Z}} \hat{G}_n^-(k) \, e^{ik\pi x}, \\ G_{2n}(x) &\equiv G_n^+(x) = \sum_{k \in \mathbb{Z}} \hat{G}_n^+(k) \, e^{ik\pi x} \end{aligned}$$

normalized as follows (the normalization conditions are written in such a way that they remain unchanged if we consider small complex valued perturbations of $V$):

$$\sum_{k \in \mathbb{Z}} \hat{G}_n^{\pm}(k) \, \hat{G}_n^{\pm}(-k) = 1, \tag{3.2}$$

$$0 = G_n^-(0) = \sum_{k \in \mathbb{Z}} \hat{G}_n^-(k), \tag{3.3}$$

$$0 = \langle G_n^-, \, G_n^+ \rangle = \sum_{k \in \mathbb{Z}} \hat{G}_n^-(k) \, \hat{G}_n^+(-k) \, . \tag{3.4}$$

The signs of $G_n^-$ and $G_n^+$ are determined in such a way that (with $' = \frac{d}{dx}$)

$$Re((G_n^-)'(0)) > 0 \tag{3.5}$$

$$Re\left( \det \begin{pmatrix} G_n^+(0) & G_n^-(0) \\ (G_n^+)'(0) & (G_n^-)'(0) \end{pmatrix} \right) > 0 \; . \tag{3.6}$$

The map $\Phi(V) := (\Phi_n(V))_{n\geq 1}$ is then defined by

$$\Phi_n(V) := \begin{pmatrix} \int_{S^1} G_n^+(x)(-\frac{d^2}{dx^2} + V - \tau_n)G_n^+(x)dx \\ \int_{S^1} G_n^-(x)(-\frac{d^2}{dx^2} + V - \tau_n)G_n^+(x)dx \end{pmatrix}. \tag{3.7}$$

According to [Ka] (cf. also [BBGK, section 4]), $\Phi : L_0^2(S^1) \to l^2(\mathbb{R}^2)$ is real analytic. Denote by $\Phi^{(N,\omega)}$ the restriction $\Phi^{(N,\omega)} := \Phi|_{H_0^{N,\omega}(S^1)}$. Note that Theorem 2 implies that $\Phi^{(N,\omega)}(H_0^{N,\omega}(S^1)) \subseteq l^2_{N,\omega}(\mathbb{R}^2)$.

Since $\Phi : L_0^2(S^1) \to l^2(\mathbb{R}^2)$ is real analytic, for any $V_0 \in L_0^2(S^1)$, there exists a neighborhood $\mathcal{U}$ of $V_0$ in $L_0^2(S^1;\mathbb{C})$ so that $\Phi$ can be extended to an analytic map, $\Phi : \mathcal{U} \to l^2(\mathbb{N};\mathbb{C}^2)$. Thus $\Phi^{(N,\omega)}$ extends to a map on $\mathcal{U} \cap H_0^{N,\omega}(S^1;\mathbb{C})$.

**Proposition 3.1.** *Assume $V_0 \in H_0^{N,\omega}(S^1;\mathbb{R})$ $(N \in \mathbb{Z}_{\geq 0}, \omega \in \mathbb{R}_{\geq 0})$.*

(i) *Then there exist a neighborhood $\mathcal{U}$ of $V_0$ in $H_0^{N,\omega}(S^1;\mathbb{C})$ and $1 \leq C < \infty$ such that $\Phi_n$ is analytic on $\mathcal{U}$ $(n \geq 1)$ and, for any $V \in \mathcal{U}$,*

$$\sum_{n\geq 1} (1+n)^{2N+2} \; e^{2n\omega} \; \left\| \Phi_n(V) - \begin{pmatrix} \frac{\hat{V}(2n)+\hat{V}(-2n)}{2} \\ \frac{\hat{V}(2n)-\hat{V}(-2n)}{2i} \end{pmatrix} \right\|^2 \; \leq \; C \; .$$

(ii) $\Phi^{(N,\omega)} \; : \; H_0^{N,\omega}(S^1) \to l^2_{N,\omega}(\mathbb{R}^2)$ *is real analytic.*

*Proof.* (ii) From (i) we conclude that $\Phi^{(N,\omega)}(\mathcal{U})$ is a bounded subset of $l^2_{N,\omega}(\mathbb{N};\mathbb{C}^2)$. Moreover, $\Phi_n$ is analytic on $\mathcal{U}$ for any $n \geq 1$. As $V_0 \in H_0^{N,\omega}(S^1)$ is arbitrary, this implies that $\Phi^{(N,\omega)} : H_0^{N,\omega}(S^1) \to l^2_{N,\omega}(\mathbb{R}^2)$ is real analytic (cf. [PT, Appendix A]).

(i) As $\Phi^{(N,\omega)}$ is the restriction of $\Phi$ and $\Phi$ is locally bounded, it suffices to find $\mathcal{U}, C, n_3 \geq 1$ such that for $V \in \mathcal{U}$

$$\sum_{n\geq n_3} (1+n)^{2N+2} \; e^{2n\omega} \; \left\| \; \Phi_n(V) \; - \; \begin{pmatrix} (\hat{V}(2n) + \hat{V}(-2n))/2 \\ (\hat{V}(2n) - \hat{V}(-2n)/2i \end{pmatrix} \right\|^2 \; \leq \; C \; . \tag{3.8}$$

To prove (3.8) we consider the cases where $\lambda_n^+ = \lambda_n^-$ and $\lambda_n^+ \neq \lambda_n^-$ separately. The statement follows from Lemma 3.2 and Lemma 3.3 below.

Let us first treat the case where $\lambda_n^+ = \lambda_n^-$ is a double eigenvalue of $-\frac{d^2}{dx^2} + V$ for $V \in H_0^{N,\omega}(S^1,\mathbb{C})$. Let $\mathcal{M} = \{n \geq 2n_2 \mid \lambda_n^+ = \lambda_n^-\}$, where $n_2$ is given by (2.26). For $n \in \mathcal{M}$ we have $z_n^+ = z_n^-$, and therefore

$$\zeta_n^+ \; = \; \zeta_n^- . \tag{3.9}$$

Together with $(\zeta_n^{\pm})^2 \; = \; A$, where $A \; = \; (\hat{V}(2n) + \beta(n, z_n^+))(\hat{V}(-2n) + \beta(-n, z_n^+))$, it follows that

$$\zeta_n^+ = \zeta_n^- = 0. \tag{3.10}$$

For $n \in \mathcal{M}$, $G_n^+$ and $G_n^-$ are of the form

$$G_n^+ = \; x_n^+ \; e^{-in\pi x} + y_n^+ \; e^{in\pi x} \; + \; \sum_{k\neq \pm n} a_{x_n^+,y_n^+,z_n^+}(k) \; e^{ik\pi x}, \tag{3.11}$$

$$G_n^- = x_n^- e^{-in\pi x} + y_n^- e^{in\pi x} + \sum_{k\neq\pm n} a_{x_n^-, y_n^-, z_n^+}(k)\, e^{ik\pi x}, \tag{3.12}$$

where $a_{x_n^\pm, y_n^\pm, z_n^+}$ is given by (2.56).

The normalization condition (3.3) leads to

$$\begin{aligned} &x_n^-(1 + \sum_{k\neq\pm n}(z_n^+ - B_n)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}(k)) \\ &\quad + y_n^-\Big(1 + \sum_{k\neq\pm n}(z_n^+ - B_n)^{-1}(\mathcal{S}^n\hat{V})_{\mathbb{Z}(n)}(k)\Big) = 0, \end{aligned} \tag{3.13}$$

which, in view of Proposition 2.11, (iv), (v), yields, as $n \geq 2n_2$ for $n \in \mathcal{M}$,

$$\begin{aligned} |x_n^-| &\leq |y_n^-|(1 + \tfrac{1}{n}\parallel V \parallel)\,\tfrac{1}{1-\frac{1}{n}\|V\|} \\ &\leq |y_n^-|\,(1 + \tfrac{4\|V\|}{n}) \leq 3|y_n^-|, \end{aligned} \tag{3.14}$$

and, similary,

$$|y_n^-| \leq |x_n^-|\,(1 + \frac{4\parallel V \parallel}{n}) \leq 3|x_n^-|\,. \tag{3.15}$$

Using (3.14) and (3.15), one obtains from the normalization condition (3.3) the estimate

$$\begin{aligned} \tfrac{2}{3}|x_n^-|^2 &\leq 2|x_n^- y_n^-| = |1 - \sum_{k\neq\pm n} a_{x_n^-, y_n^-, z_n^+}(k) a_{x_n^-, y_n^-, z_n^+}(-k)| \\ &\leq 1 + \parallel a_{x_n^-, y_n^-, z_n^+} \parallel^2 \leq 1 + \frac{|x_n^-| + |y_n^-|}{2}\,\frac{\parallel V \parallel}{n} \leq 1 + 2|x_n^-|\,\frac{\parallel V \parallel}{n}\,. \end{aligned}$$

It follows that for $n \geq 3n_2$,

$$|x_n^-| \leq \sqrt{\frac{16}{10}} + \frac{2}{3}\,\frac{\parallel V \parallel}{n} \leq 2. \tag{3.16}$$

Similarly, one can show that for $n \geq 3n_2$

$$|y_n^-| \leq \sqrt{\frac{16}{10}} + \frac{2}{3}\,\frac{\parallel V \parallel}{n} \leq 2. \tag{3.17}$$

Therefore, (3.13) leads to, using Proposition 2.11 (vi),

$$|x_n^- + y_n^-| \leq \frac{4}{n} \parallel V \parallel, \tag{3.18}$$

and (3.2) implies, for $n \geq 3n_2$, using Proposition 2.11 (iii),

$$|2x_n^- y_n^- - 1| \leq (\frac{4\parallel V \parallel}{n})^2 \leq \frac{4\parallel V \parallel}{n}. \tag{3.19}$$

Estimates (3.18) and (3.19) imply, with (3.5), $n \in \mathcal{M}$, $n \geq 3n_2$,

$$|x_n^- - \frac{i}{\sqrt{2}}| \leq \frac{10\parallel V \parallel}{n}; \quad |y_n^- + \frac{i}{\sqrt{2}}| \leq \frac{10\parallel V \parallel}{n}. \tag{3.20}$$

Using (3.2) and (3.4) and the above estimates, we conclude, by similar computations, that, for $n \in \mathcal{M}$ with $n \geq 20n_2$,

$$
\begin{aligned}
|x_n^+| &\leq 3, \quad |x_n^+ - \frac{1}{\sqrt{2}}| \leq \frac{200 \parallel V \parallel}{n}; \\
|y_n^+| &\leq 3, \quad |y_n^+ - \frac{1}{\sqrt{2}}| \leq \frac{200 \parallel V \parallel}{n}.
\end{aligned}
\tag{3.21}
$$

For $n \in \mathcal{M},\ n \geq 20n_2$, one obtains

$$
\begin{aligned}
(-\frac{d^2}{dx^2} + V - \tau_n)\ G_n^+ = (\lambda_n^+ - \tau_n)\ G_n^+ + (\hat{V}(-2n) + \beta(-n, z_n^+))\ y_n^+\ e^{-in\pi x} \\
+ (\hat{V}(2n) + \beta(n, z_n^+))\ x_n^+\ e^{in\pi x},
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\langle G_n^+,\ (-\frac{d^2}{dx^2} + V - \tau_n) G_n^+ \rangle \\
&\quad = (\hat{V}(2n) + \hat{V}(-2n)) x_n^+ y_n^+ \ + \ x_n^+ y_n^+ (\beta(n, z_n^+) + \beta(-n, z_n^+)) \\
&\quad = \frac{1}{2}(\hat{V}(2n) + \hat{V}(-2n)) + \ l^2_{N+1,\omega}(n)
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle G_n^-,\ (-\frac{d^2}{dx^2} + V - \tau_n)\ G_n^+ \rangle \\
&\quad = x_n^- y_n^+ \hat{V}(-2n) + y_n^- x_n^+\ \hat{V}(2n) + x_n^- y_n^+ \beta(-n, z_n^+) + y_n^- x_n^+ \beta(n, z_n^+) \\
&\quad = \frac{i}{2}\ (\hat{V}(-2n) - \hat{V}(2n)) \ + \ l^2_{N+1,\omega}\ (n),
\end{aligned}
$$

where, by abuse of notation, we mean by $(l^2_{N+1,\omega}(n))_{n\geq 1}$ an element in $l^2_{N+1,\omega}$ uniformly bounded for $V \in \mathcal{U}$, where $\mathcal{U}$ is a sufficiently small neighborhood of $V_0 \in H_0^{N,\omega}(S^1, \mathbb{R})$ in $H_0^{N,\omega}(S^1; \mathbb{C})$. Using the above estimates, one obtains

**Lemma 3.2.** *Assume that $V_0 \in H_0^{N,\omega}(S^1, \mathbb{R})$. Then there exist a neighborhood $\mathcal{U}$ of $V_0$ in $H^{N,\omega}(S^1; \mathbb{C}), n_3 \geq n_2$ and $1 \leq C < \infty$ so that for $V \in \mathcal{U}$,*

$$
\sum_{\substack{n \geq n_3 \\ \lambda_n^+ = \lambda_n^-}} (1+n)^{2N+2}\ e^{2n\omega}\ \ \parallel \Phi_n(V) - \begin{matrix} (\hat{V}(2n) + \hat{V}(-2n))/2 \\ (\hat{V}(2n) - \hat{V}(-2n))/2i \end{matrix} \parallel^2 \leq C.
$$

Let us now consider those $n \geq n_2$ with $\lambda_n^+ \neq \lambda_n^-$. Denote by $f_n^{\pm}$ eigenfunctions of $\lambda_n^{\pm}$ normalized so that

$$
\sum_{k \in \mathbb{Z}} \hat{f}_n^{\pm}(k)^2 = 1\ .
$$

Then $G_n^+$ and $G_n^-$ are linear combinations of $f_n^+$ and $f_n^-$,

$$
\begin{aligned}
G_n^+ &= \alpha_n^+ f_n^+ + \alpha_n^- f_n^- \\
&= x_n^+\ e^{-in\pi x} \ + \ y_n^+\ e^{in\pi x} \ + \sum_{k \neq \pm n} b_n^+(k)\ e^{ik\pi x}
\end{aligned}
\tag{3.22}
$$

with

$$b_n^+ \ := \ \alpha_n^+ \hat{f}_n^+(-n)(z_n^+ - B_n)^{-1}\mathcal{S}^n\hat{V} \ + \ \alpha_n^- \hat{f}_n^-(-n)(z_n^- - B_n)^{-1}\mathcal{S}^n\hat{V}$$
$$+\alpha_n^+ \hat{f}_n^+(n)(z_n^+ - B_n)^{-1}\mathcal{S}^{-n}\hat{V} + \alpha_n^- \hat{f}_n^-(n)(z_n^- - B_n)^{-1}\mathcal{S}^{-n}\hat{V}$$

and

$$(3.23) \qquad \begin{aligned} G_n^- &= \beta_n^+ f_n^+ + \beta_n^- f_n^- \\ &= x_n^- \, e^{-in\pi x} \ + \ y_n^- \, e^{in\pi x} \ + \ \sum_{k \neq \pm n} b_n^-(k) \, e^{ik\pi x} \end{aligned}$$

with $b_n^-$ given by an expression similar to $b_n^+$.

For $V_0 \in L_0^2(S^1;\mathbb{R})$ there exist a neighborhood $\mathcal{U}$ of $V_0$ in $L_0^2(S^1;\mathbb{C})$ and $n_3 \geq n_2$ so that for $n \geq n_3$ (cf. [BBGK, Lemma 4.10], [Ka, Proposition 8])

$$(3.24^+) \qquad x_n^+ \ = \ \frac{1}{\sqrt{2}} \ + \ O(\frac{1}{n}) \ , \ y_n^+ \ = \ \frac{1}{\sqrt{2}} \ + \ O(\frac{1}{n}),$$

$$(3.24^-) \qquad x_n^- \ = \ \frac{i}{\sqrt{2}} \ + \ O(\frac{1}{n}) \ , \ y_n^- \ = \ -\frac{i}{\sqrt{2}} \ + \ O(\frac{1}{n}),$$

$$(3.25^+) \qquad f_n^+(x) \ = \ \frac{e^{-i\theta_n}}{\sqrt{2}} \, e^{-in\pi x} \ + \ \frac{e^{i\theta_n}}{\sqrt{2}} \, e^{in\pi x} \ + O(\frac{1}{n}),$$

$$(3.25^-) \qquad f_n^-(x) \ = \ i\frac{e^{-i\theta_n}}{\sqrt{2}} \, e^{-in\pi x} \quad - i\frac{e^{i\theta_n}}{\sqrt{2}} \, e^{in\pi x} \ + O(\frac{1}{n}) \ .$$

As $\lambda_n^+ \neq \lambda_n^-$, the normalization conditions for $G_n^\pm$ imply

$$(\alpha_n^+)^2 + (\alpha_n^-)^2 \ = \ 1, \ (\beta_n^+)^2 + (\beta_n^-)^2 = 1, \ \alpha_n^+\beta_n^+ + \alpha_n^-\beta_n^- = 0,$$

which leads to

$$(3.26) \qquad \alpha_n^+ \ = \ \frac{e^{i\theta_n} + e^{-i\theta_n}}{2} \ + \ O(\frac{1}{n}) \ , \ \alpha_n^- \ = \ \frac{e^{i\theta_n} - e^{-i\theta_n}}{2i} \ + \ O(\frac{1}{n}),$$

$$(3.27) \qquad \beta_n^+ = -\frac{e^{i\theta_n} - e^{-i\theta_n}}{2i} \ + \ O(\frac{1}{n}) \ , \ \beta_n^- \ = \ \frac{e^{i\theta_n} + e^{-i\theta_n}}{2} \ + \ O(\frac{1}{n}).$$

When written in Fourier space, $(-\frac{d^2}{dx^2} + V - \tau_n)G_n^+$ takes the form (cf. (2.3))

$$(3.28) \qquad \begin{pmatrix} - \quad x_n^+(z_n^+ + z_n^-)/2 \ + \ y_n^+\hat{V}(-2n) \ + \ \langle \mathcal{S}^n\mathcal{J}\hat{V}, b_n^+\rangle \\ x_n^+\hat{V}(2n) \ - \ y_n^+(z_n^+ + z_n^-)/2 \ + \ \langle \mathcal{S}^{-n}\mathcal{J}\hat{V}, b_n^+\rangle \\ x_n^+\mathcal{S}^n\hat{V} \ + \ y_n^+\mathcal{S}^{-n}\hat{V} \ + \ (B_n - (z_n^+ + z_n^-)/2)b_n^+ \end{pmatrix}.$$

The terms appearing in this expression are discussed separately. Since $\alpha(n,z) = \alpha(-n,z)$, one obtains

$$\begin{aligned}
\langle \mathcal{S}^n \mathcal{J}\hat{V}, b_n^+\rangle &= \alpha_n^+ \hat{f}_n^+(-n)\ \alpha(n,z_n^+) \ +\ \alpha_n^- \hat{f}_n^-(-n)\ \alpha(n,z_n^-) \\
&\quad + \alpha_n^+ \hat{f}_n^+(n)\ \beta(-n,z_n^+) \ +\ \alpha_n^- \hat{f}_n^-(n)\beta(-n,z_n^-) \\
&= x_n^+\ \frac{\alpha(n,z_n^+) + \alpha(n,z_n^-)}{2} \ +\ x_n^+\ \frac{\alpha(n,z_n^+) - \alpha(n,z_n^-)}{2} \\
&\quad +\ \alpha_n^- \hat{f}_n^-(-n)(\alpha(n,z_n^+) - \alpha(n,z_n^-)) \\
&\quad + \alpha_n^+ \hat{f}_n^+(n)\ \beta(-n,z_n^+) \ +\ \alpha_n^- \hat{f}_n^-(n)\beta(-n,z_n^-).
\end{aligned} \tag{3.29}$$

Taking into account that $-\zeta_n^\pm = -z_n^\pm + \alpha(n,z_n^\pm)$ and thus

$$-\frac{z_n^+ + z_n^-}{2} \ +\ \frac{\alpha(n,z_n^+) + \alpha(n,z_n^-)}{2} \ =\ -\frac{\zeta_n^+ + \zeta_n^-}{2},$$

the first component in (3.28) can be written as

$$-x_n^+\ (\frac{z_n^+ + z_n^-}{2}) \ +\ y_n^+ \hat{V}(-2n) + \langle \mathcal{S}^n \mathcal{J}\hat{V}, b_n^+\rangle \ =\ \hat{V}(-2n) y_n^+ \ +\ g_n^+(-n), \tag{3.30}$$

where

$$\begin{aligned}
g_n^+(-n) \quad &= \quad (x_n^+ + \alpha_n^- \hat{f}_n^-(-n))(\alpha(n,z_n^+) - \alpha(n,z_n^-)) - x_n^+\ \tfrac{\zeta_n^+ + \zeta_n^-}{2} \\
&\quad + \alpha_n^+ \hat{f}_n^+(n)\beta(-n,z_n^+) \ +\ \alpha_n^- \hat{f}_n^-(n)\beta(-n,z_n^-).
\end{aligned} \tag{3.31}$$

In view of (3.24)–(3.27), Lemma 2.4(ii), Proposition 2.7 and Theorem 2.10 we conclude that $g_n^+(-n) = l^2_{N+1,\omega}(n)$, uniformly bounded for $V$ in a sufficiently small neighborhood $\mathcal{U}$ of $V_0$ in $H_0^{N,\omega}(S^1,\mathbb{C})$.

The second component in (3.28) is analyzed similarly, to yield

$$x_n^+ \hat{V}(2n) - y_n^+(z_n^+ + z_n^-)/2 \ + \langle \mathcal{S}^{-n} \mathcal{J}\hat{V}, b_n^+\rangle = \hat{V}(2n) x_n^+ + g_n^+(n), \tag{3.32}$$

where

$$\begin{aligned}
g_n^+(n) &= (-y_n^+ + \alpha_n^+ \hat{f}_n^+(n))\ \frac{\alpha(n,z_n^+) - \alpha(n,z_n^-)}{2} \ -\ y_n^+ \frac{\zeta_n^+ + \zeta_n^-}{2} \\
&\quad + \alpha_n^+ \hat{f}_n^+(-n)\beta(n,z_n^+) + \alpha_n^- \hat{f}_n^-(-n)\beta(n,z_n^-),
\end{aligned} \tag{3.33}$$

and we again conclude that $g_n^+(-n) = l^2_{N+1,\omega}(n)$.

Finally we analyze the third component in (3.28):

We compute (with $\gamma_n := \lambda_n^+ - \lambda_n^-$)

$$
\begin{aligned}
(B_n - \frac{z_n^+ + z_n^-}{2})b_n^+ = &-(\alpha_n^+ \hat{f}_n^+(-n) \ + \ \alpha_n^- \hat{f}_n^-(-n))\mathcal{S}^n \hat{V} \\
&- (\alpha_n^+ \hat{f}_n^+(n) \ + \ \alpha_n^- \hat{f}_n^-(-n))\mathcal{S}^{-n} \hat{V} \\
&+ \alpha_n^+ \hat{f}_n^+(-n) \ \frac{\gamma_n}{2}(z_n^+ - B_n)^{-1}\mathcal{S}^n \hat{V} \\
&+ \alpha_n^- \hat{f}_n^-(-n) \ \frac{\gamma_n}{2}(z_n^- - B_n)^{-1}\mathcal{S}^n \hat{V} \\
&+ \alpha_n^+ \hat{f}_n^+(n) \ \frac{\gamma_n}{2}(z_n^+ - B_n)^{-1}\mathcal{S}^{-n} \hat{V} \\
&+ \alpha_n^- \hat{f}_n^-(n) \ \frac{\gamma_n}{2}(z_n^- - B_n)^{-1}\mathcal{S}^{-n} \hat{V} \ .
\end{aligned}
$$

Thus

$$
\begin{aligned}
(3.34) \qquad & x_n^+ \mathcal{S}^n \hat{V} + y_n^+ \mathcal{S}^{-n} \hat{V} + (B_n - \frac{z_n^+ + z_n^-}{2})b_n^+ \\
= \quad & \alpha_n^+ \hat{f}_n^+(-n) \ \tfrac{\gamma_n}{2}(z_n^+ - B_n)^{-1}\mathcal{S}^n \hat{V} \\
& + \alpha_n^- \hat{f}_n^-(-n) \ \tfrac{\gamma_n}{2}(z_n^- - B_n)^{-1}\mathcal{S}^n \hat{V} \\
& + \alpha_n^+ \hat{f}_n^+(n) \ \tfrac{\gamma_n}{2}(z_n^+ - B_n)^{-1}\mathcal{S}^{-n} \hat{V} \\
& + \alpha_n^- \hat{f}_n^-(n) \ \tfrac{\gamma_n}{2}(z_n^- - B_n)^{-1}\mathcal{S}^{-n} \hat{V} \\
= \quad & l^2_{N+1,\omega}(n).
\end{aligned}
$$

Combining (3.30)-(3.34), we obtain, for $n \geq n_3$ with $\lambda_n^+ \neq \lambda_n^-$,

$$
\begin{aligned}
\langle G_n^+, (-\tfrac{d^2}{dx^2} + V - \tau_n) G_n^+ \rangle \ &= \ \hat{V}(-2n)(y_n^+)^2 \ + \ \hat{V}(2n)(x_n^+)^2 \ + \ l^2_{N+1,\omega}(n), \\
\langle G_n^-, (-\tfrac{d^2}{dx^2} + V - \tau_n) G_n^+ \rangle \ &= \ \hat{V}(-2n) x_n^- y_n^+ + \hat{V}(2n) y_n^- x_n^+ + l^2_{N+1,\omega}(n).
\end{aligned}
$$

In view of ($3.24^{\pm}$) one then concludes the following:

**Lemma 3.3.** *Assume $V_0 \in H^{N,\omega}(S^1;\mathbb{R})$. Then there exist a neighborhood $\mathcal{U}$ of $V_0$ in $H^{N,\omega}(S^1;\mathbb{C})$, $n_3 \geq n_2$ and $1 \leq C < \infty$ such that for any $V \in \mathcal{U}$ and $n \geq n_3$*

$$
\sum_{\substack{n \geq n_3 \\ \lambda_n^+ \neq \lambda_n^-}} (1+n)^{2N+2} \ e^{2n\omega} \left\| \Phi_n(V) - \begin{pmatrix} \frac{\hat{V}(2n) + \hat{V}(-2n)}{2} \\ \frac{\hat{V}(2n) - \hat{V}(-2n)}{2i} \end{pmatrix} \right\|^2 \leq C.
$$

**Proposition 3.4.** *The map $\Phi^{(N,\omega)} : H_0^{N,\omega}(S^1) \to l^2_{N,\omega}(\mathbb{R}^2)$ has the following properties:*

(i) $\Phi^{(N,\omega)}$ *is bijective and real analytic;*

(ii) $(\Phi^{(N,\omega)})^{-1}$ *is real analytic;*

*Proof.* As $\Phi^{(N,\omega)} = \Phi\big|_{H_0^{N,\omega}(S^1)}$ and $\Phi : L_0^2(S^1) \to l^2(\mathbb{R}^2)$ is bijective and bianalytic, we conclude that $\Phi^{(N,\omega)}$ is one-to-one and, for $V \in H_0^N(S^1)$, $d_V\Phi^{(N,\omega)}$ is also one-to-one. From Corollary 3, stated in the introduction, and the fact that $\Phi$ is onto we conclude that $\Phi^{(N,\omega)}$ is onto. By Proposition 3.1, $\Phi^{(N,\omega)}$ is real analytic. To prove statement (ii), it suffices to show that $d_V\Phi^{(N,\omega)}$ is onto for an arbitrary element $V$ in $H_0^{N,\omega}(S^1)$. By (i) and the Fredholm alternative it suffices to prove that $d_V\Phi^{(N,\omega)} = A + K$, where $A : H_0^{N,\omega}(S^1) \to l^2_{N,\omega}(\mathbb{R}^2)$ is a linear isomorphism and $K$ is a compact operator. This follows from Proposition 3.1. □

Next we consider the map $\Lambda : L_0^2(S^1) \to l^2_{1/2}(\mathbb{R}^2)$, defined in [BBGK] by $\Lambda(V) = (\Lambda_n(V))_{n\geq 1}$ with $\Lambda_n(V) = \xi_n(V)\Phi_n(V)$. Here

$$\xi_n(V)^2 = \frac{2\mathcal{I}_n(V)}{\gamma_n(V)/2}$$

and $\mathcal{I}_n(V)(n \geq 1)$ denote the action variables of KdV with respect to the Gardner bracket ([FM]; cf. [BBGK])

$$\mathcal{I}_n(V) = \frac{2}{\pi} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \mu \frac{\dot{\triangle}(\mu)}{(\triangle(\mu) - 4)^{1/2}}\, d\mu\ .$$

Define $\Lambda^{(N,\omega)} := \Lambda\big|_{H_0^{N,\omega}(S^1)}$. From the asymptotics, valid uniformly on sets of potentials bounded in $L_0^2(S^1;\mathbb{C})$.

$$\xi_n = \frac{1}{\sqrt{n\pi}}\,(1 + O(\frac{\log n}{n})) \tag{3.35}$$

(cf. [BBGK]) we conclude that $\Lambda^{(N,\omega)}(H_0^{N,\omega}(S^1)) \subset l^2_{N+\frac{1}{2},\omega}(\mathbb{R}^2)$.

**Proposition 3.5.** *The map* $\Lambda^{(N,\omega)} : H_0^{N,\omega}(S^1) \to l^2_{N+\frac{1}{2},\omega}(\mathbb{R}^2)$ *has the following properties:*

(i) $\Lambda^{(N,\omega)}$ *is bijective and real analytic;*
(ii) $(\Lambda^{(N,\omega)})^{-1}$ *is real analytic;*
(iii) *for any* $0 \leq \varepsilon < 1/2$,

$$\Lambda_n^{(N,\omega)}(V) = \frac{1}{\sqrt{n\pi}}\Big(\frac{\hat{V}(2n) + \hat{V}(-2n)}{2},\ \frac{\hat{V}(2n) - \hat{V}(-2n)}{2i}\Big) + l^2_{N+1+\varepsilon,\omega}(n)$$

*uniformly on sets of potentials bounded in* $H_0^{N,\omega}(S^1;\mathbb{C})$.

*Proof.* We argue as in the proof of Proposition 3.4 to conclude from [BBGK, section 2] that $\Lambda^{(N,\omega)}$ is bijective and real analytic and that $d_V\Lambda^{(N,\omega)}$ is one-to-one for any $V \in H_0^{N,\omega}(S^1)$. Statement (iii) follows from (3.35) and Proposition 3.1. To prove statement (ii) it suffices to prove that $d_V\Lambda^{(N,\omega)}$ is onto for any $V$ in $H_0^{N,\omega}(S^1)$.

Write

$$d_V\Lambda_n(W) = \xi_n(V)\, d_V\Phi_n(W) + d_V\xi_n(W)\Phi_n(V) \tag{3.36}$$

and introduce, for $V$ fixed,

$$A := (A_n)_{n\geq 1} : H_0^{N,\omega}(S^1) \to l^2_{N+\frac{1}{2},\omega}(\mathbb{R}^2)$$

with $A_n(W) := \frac{1}{\sqrt{n\pi}} d_V \Phi_n(W)$. By Proposition 3.4, $A$ is a linear isomorphism. Moreover, by [BBGK, Lemma 4.18], for any $0 \le \varepsilon < 1/2$,

$$d_V \, \xi_n(W) = O(\frac{1}{n^{1+\varepsilon}}) \, \| W \|_{L^2} . \tag{3.37}$$

Substituting (3.35) and (3.37) into (3.36), we conclude that, for any $0 \le \varepsilon < 1/2$, there exists $C_\varepsilon > 0$ such that

$$\| \left( d_V \Lambda_n(W) - A_n(W) \right)_{n \ge 1} \|_{l^2_{N+1+\varepsilon,\omega}} \le C_\varepsilon \| W \|_{L^2} .$$

Estimate (3.21) implies that $d_V \Lambda^{(N,\omega)} = A + K$, where $K$ is compact. By the Fredholm alternative and the fact that $d_V \Lambda^{(N,\omega)}$ is one-to-one, we conclude that $d_V \Lambda^{(N,\omega)}$ is onto. This implies statement (ii). □

*Proof of Theorem 1.* Theorem 1 follows from Proposition 3.5. □

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